Physics of Stellar Interiors

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June 25, 2002

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Chapter 1

Introduction

1.1 Preface

Present text is a draft and not a finished manuscript of lecture notes. The hope is that it would prove useful for the students and that in the course of time it would evolve to a more decent form.

On one hand theory of stellar structure is a mature theory in a sense that it is well rooted in observations and the basic physics is well understood. On the other hand it still offers many fascinating and ill understood problems, concerning of star formation, late stages of evolution, hydrodynamics of pulsations and explosions, astro-seismology and evolution of the primordial (III) population of stars. Several excellent textbooks treat stellar structure in great detail (Kavaler, Kippenhahn & Weigert, Cox & Giuli). They ought be studied by any student and researcher planning to specialize in this field. Present lecture notes are intended to offer a first glimpse of the theory to physics and astronomy students. By the time these students choose to read stellar structure they are versed in mathematics of physics. For this reason and to spend more time on physics skip many exact analytical derivations. Main stress is on explanation of the underlying physics. We try to demonstrate how complex processes may be analyzed using simple approximations and estimates. The ability to do approximate estimates is vital in doing research, yet students are rarely taught them. Showing that this is right approach to teaching of astrophysics we own to D. Rose and his book on Astrophysics.

Textbooks

Cox & Giuli, ***

Kawaler, S., ***, Springer

Table 1.1: Characteristics of the Sun

Parameter	Symbol	Value [cgs]
Mass	$M_\odot$	$2 \cdot 10^{33} g$
Radius	$R_\odot$	$7 \cdot 10^{10} cm$
Luminosity	$L_\odot$	$4 \cdot 10^{33} erg/s$
Surface temperature	T_e	$6000K$
Composition	X	0.7 H, 0.28 He & 0.02 CNO by weight

Kippenhahn, R, & Weigert, W., ***, Springer

Rose, D, *** Astrophysics

1.2 Observations of stars

1.2.1 Characteristics of the Sun

Observations and simple calculations yield several global characteristics of stars. Their values for our Sun are fairly typical for many stars. Mass $M_\odot$ is estimated from size $A_\dagger$ and period of the orbit of Earth, using Kepler law. Radius $R_\odot$ is determined from angular radius and $A_\dagger$. Similarly estimates of luminosity $L_\odot$, i.e. power of radiation follow from the total (bolometric) flux observed from Earth, i.e. solar constant, times area of a sphere of radius $A_\dagger$. Surface temperature $T_{e\odot}$ and chemical composition X follow respectively from fitting continuum and line profiles of solar spectrum with the atmosphere models. Rough values of these parameters are listed in Table 1.1.

1.2.2 Stability of the Sun

Geologic evidence indicates old age of the Solar System, exceeding $4 \cdot 10^9 y$ (years). Development of life during much of this time required stable temperature on the Earth: below $0C$ oceans would freeze, above $50C$ most proteins would decompose. This in turn indicates stable luminosity of our Sun during most of life of the Earth. We shall demonstrate that balance of sources and losses of energy may be reached in time short to this observation. It means that the Sun has reached the energy (thermal) balance.

No such old evidence exists for size of the Sun. Measurements of solar diameter started well after invention of the telescope and cover no more than two centuries. Unfortunately they are marred by systematic errors, so that according to them constancy of solar radius is guaranteed for no more then several thousands of years. In this respect very helpful is near coincidence of the solar and lunar angular diameters. Due to slight eccentricities of respective orbits Moon and Sun distances vary enough to alternate order of their angular distances causing annular or total solar eclipses for Moon angular diameter respectively small or large compared to Sun angular diameter. Now very slight increase (decrease) of the solar radius would turn extra total (annular) eclipses. In this respect ancient astronomical records from Egypt and China combined with modern precise orbital calculations of the Solar System exclude any significant change in the solar radius in the last 10^4 years. Short as this time is, it is much longer than the dynamical time scale, i.e. time of restoration of balance of gravity and pressure forces. Thus to maintain solar radius fixed, gas pressure and gravity within the Sun must remain in the mechanical (hydrostatic) equilibrium.

1.2.3 Correlation of stellar parameters

Research of stars grouped in globular and open clusters and in the Milky Way reveals that parameters of most stars are highly correlated. Hertzsprung and Russell (hereafter H-R) demonstrated that in the luminosity-temperature (L - T) plot most stars lay along one line called a main sequence. Other sequences are observed in the mass-luminosity (M - L) and mass-radius (M - R) plots. These facts combined mean that in the many dimensional space of stellar parameters most stars aggregate along the curve corresponding to the main sequence. Every point on the curve yields as coordinates a set of valid parameter values for a star. Value of one coordinate suffices to specify uniquely location on a curve. In case of the main sequence knowledge of value of just one parameter (e.g. mass or temperature) specifies all other stellar properties, i.e. essentially the whole structure of a star. We must caution that our conclusion holds for the stars formed from gas clouds of the same chemical composition, i.e. for $X = const$. This is likely true for most stars in the groups listed at the start of the present section.

Chapter 2

Mechanical equilibrium

2.1 Hydrostatic equilibrium

2.1.1 Continuity equation

Stars are gas clouds kept in mechanical equilibrium by balance of the gas pressure and own gravity. This equilibrium is called hydrostatic equilibrium. Pascal and Newton laws guarantee isotropy of pressure and gravity forces, respectively. Hence in absence of other forces stars adopt spherical shape.

One way of description of the distribution of mass within a spherical star is by listing density ρ as function of distance r from the star center, $\rho = \rho(r)$. Another way is by listing stellar mass content $m = m(r)$ within given radius r from the star center. Increase of distance by dr corresponds to growth of mass by the mass dm contained in a thin spherical layer of radius r and thickness dr. Its volume is $4\pi r^2 dr$ so

$$dm = 4\pi r^2 \rho dr \tag{2.1}$$

This is continuity equation reflecting mass conservation principle. In these terms for the total mass of the star M and its radius R we obtain $M = m(R)$.

2.1.2 Gravity field

The gravity field in a spherical star has simple structure. According a theorem proven by Newton, at distance r from the center, gravity forces from masses outside radius r cancel and the remaining attraction is such as for the point mass m located in the center. The corresponding gravity accelera tion is

$$g = Gm/r^2. \tag{2.2}$$

This equation yields gravity field for spherical stars.

Table 2.1: Boundary conditions

Location	Radius	Mass	Pressure
Center	$\boxed{r=0}$	$\boxed{m=0}$	$P \equiv P_c$
Surface	$r \equiv R$	$m \equiv M$	$\boxed{P \approx 0}$

Defining potential Φ such that $d\Phi/dr \equiv g$ we obtain $(d/dr)(r^2 d\Phi/dr) = (d/dr)gr^2 = Gdm$. Substituting dm from the continuity equation we obtain Poisson equation for Φ:

$$\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G r^2 \rho \tag{2.3}$$

Poisson equation easily generalizes in 3-dimensions for asymmetric mass distributions of rotating and tidally distorted stars for which Newton theorem no longer holds.

2.1.3 Hydrostatic equilibrium equation

Within a thin spherical layer in the mechanical equilibrium pressure P has to grow by the weight of the gas column of length dr and unit surface cross-section $dP = -\rho 1 dr g$, where g is gravity acceleration at distance r from the center of star. According a theorem proven by Newton, at distance r from the center of a spherical mass distribution, gravity forces from masses outside radius r cancel and the remaining attraction is such as for the point mass m located in the center. In our notation $g = Gm/r^2$ and

$$dP = -\frac{Gm}{r^2}\rho dr \tag{2.4}$$

This is the hydrostatic equilibrium equation.

2.1.4 Boundary conditions

To obtain unique solution of differential equations one requires boundary conditions. The boundary conditions for stellar structure equations are defined at two different locations: at the center and on the surface of a star. They are listed in the Table 2.1. Three boundary conditions are framed. Other expressions are simply definitions of new parameters: central pressure, radius and mass of the star P_c, R and M. Two differential equations of stellar structure, namely the continuity and hydrostatic equilibrium equations contain 3 unknown functions of r: m, P and ρ. At least one more equation is required to close the system.

2.1.5 Hydrostatic equilibrium equation

Thy system of the differential equations (2.1) and (2.1) is supplemented by an equation of state. An equation of state is an algebraic equation relating different state parameters, i.e. parameters determining local state of gas, such us $P, \rho, *, \cdots$. Its most general form is

$$0 = f(P, \rho, * \cdots). \tag{2.5}$$

An equation of state closes the system of stellar structure equations if all its parameters $(*)$ except P and ρ are known. Such equations are called barotropic equations. By no means all equations of state are barotropic. Best known equation of state valid for gas in a near perfect state depends on temperature T, which is a priori unknown. The equations of state encountered in astrophysics usually assume simpler form than (2.5): $P = f(\rho, * \cdots)$.

2.2 Simple equations of state

To proceed further we need specific equations of state. Here we list some of them. Later we devote a whole chapter for their derivation and discussion.

2.2.1 Perfect gas

The equation of state for perfect gas of particles of mass μ at temperature T is

$$\frac{P_g}{\rho} = \tfrac{kT}{\mu} = (\gamma - 1)c_V T \quad \text{where} \tag{2.6}$$

$$U_g = c_V T \quad \text{is internal energy} \tag{2.7}$$

$$\gamma = \tfrac{c_P}{c_V} = \tfrac{c_V + \mathcal{R}}{c_V} \quad \text{is adiabatic exponent} \tag{2.8}$$

$$k = \tfrac{\mathcal{R}}{N} \quad \text{is Boltzmann constant} \tag{2.9}$$

$\mathcal{R}$ is gas constant, N is number of particles per mole (Avogadro constant), c_P and c_V are specific heats at constant pressure and volume, respectively. Note that our equation multiplied by $N\mu$ yields on the left hand side PV where V is volume of one mole of gas.

2.2.2 Radiation

According to Stefan-Boltzmann law flux, i.e. energy emitted from unit surface of a perfect black body is $F = \sigma T^4$. Energy emitted from unit surface into spherical angle $d\Omega = 2\pi \sin\theta d\theta$ is $I d\Omega$, where I is intensity of radiation. By definition

Table 2.2: Specific polytropes

Name	γ	n	Invariant
isobar	0	-1	P
isotherm	1	∞	T
adiabate	c_P/c_V	$1/(\gamma-1)$	heat
isochore	∞	0	ρ

black body radiation is emitted in thermodynamic equilibrium where temperature everywhere is the same and hence radiation is isotropic. In terms of I flux is

$$F = \int I \cos\theta d\Omega = \pi I \tag{2.10}$$

where $\cos\theta$ accounts for projection of the unit surface onto direction of emission and integration extends over a half sphere. Momentum of a photon is its energy divided by light velocity c: $p = h\nu/c$. So momentum flux in a given direction is energy flux divided by c. Normal component of the momentum flux is by another factor $\cos\theta$ less than the momentum flux. Pressure is force per unit surface. By virtue of Newton second law force is change of momentum per unit time. For particles bouncing from a wall it is twice the momentum flux:

$$P_r = 2\int \frac{I}{c} \cos^2\theta d\Omega = \frac{4\pi I}{3c} \tag{2.11}$$

Substituting F we obtain

$$P_r = \frac{4\sigma}{3c} T^4 \tag{2.12}$$

Photons leaving each second unit surface in direction of $d\Omega$ occupy volume $1 \cdot c$. The corresponding energy density is $I d\Omega/c = F d\Omega/\pi c$. The total energy density U_r of photons moving in the full spherical angle 4π is

$$U_r = \frac{4\sigma}{c} T^4 = \frac{P_r}{3} \tag{2.13}$$

2.2.3 Polytropic equation of state

The polytropic equation of state is by definition

$$P = K\rho^\gamma = K\rho^{1+1/n} \tag{2.14}$$

where K, γ and n are constant. The polytropes for special values of γ correspond to specific gas processes (Table 2.2). For a star with constant density, $\gamma = \infty$ we obtain average density

$$< \rho >= 3M/4\pi R^3 \tag{2.15}$$

For a star with fixed radiation and gas pressure ratio we obtain $const = P_r/P_g \sim T^4/\rho T$, hence $T \sim \rho^{1/3}$ and finally

$$P_r \sim P_g \sim \rho^{4/3}, \quad \text{i.e.} \quad \gamma = 4/3. \tag{2.16}$$

2.3 Estimates of stellar properties

2.3.1 Conditions at center

Rough estimates of values of stellar parameters follow from substitution of infinite differentials by finite differences of surface and central values. In this way from the hydrostatic equilibrium equation we obtain:

$$(0 - P_c) \sim -\frac{GM/2}{(R/2)^2}\frac{3M}{4\pi R^3} \tag{2.17}$$

On R.H.S. for m, r and ρ we substituted their corresponding average values $M/2$, $R/2$ and Eq. (2.15). By omitting all factors of order unity (e.g. 2 and π) we obtain

$$P_c \sim \frac{GM^2}{R^4} \sim GM^{2/3}\rho^{4/3} \sim 10^{14} dyne/cm^2. \tag{2.18}$$

where the final number corresponds to the Sun. The similarity sign in Eq. (2.18) indicates that P_c scales roughly proportional to the R.H.S., so long as shape of the distribution of the state parameters inside a star remains the same and only scale of variation changes. The proportionality constant remains of order unity, for smooth distributions justifying substitution of finite differences for the infinite differentials. Substituting P_c into perfect gas equation of state we obtain for the central temperature of a star:

$$T_c \sim \frac{G\mu}{k}\frac{M}{R} \sim\sim 10^7 K. \tag{2.19}$$

Two interesting consequences follow from these estimates. First, the central temperature is so high, $kT_c \sim 1keV$ that sporadic nuclear reactions are possible. Second, since the central temperature exceeds the surface one by many orders of magnitude, $T_c \gg T_e$, so does pressure $P_c \gg P_e \approx 0$ for any non-pathological γ. Hence we expect little error in the solutions by setting $P_e = 0$ exactly.

2.3.2 Energy and virial theorem

The gravitational energy E_G of a star corresponds roughly to the energy of two masses $m = M/2$ acting from distance $r = R/2$:

$$E_G \sim -\frac{GM^2}{R} \tag{2.20}$$

Again we neglected all factors of order unity. The average thermal energy per particle is of order of kT. Indeed this follows from the thermodynamic definition of T as measure of the kinetic energy of particles. The thermal energy E_T of a star consisting of M/μ particles is

$$E_T \sim \frac{M}{\mu} kT \sim \frac{GM^2}{R} \sim E_G \tag{2.21}$$

The final form of the R.H.S. was obtained by substitution of Eq. (2.20). Note that our derivation does not mean that $E_T = E_G$. On the contrary, the two energies remain generally different but scale with mass and radius M and R in the same way. Generally a relation of different forms of energy of a configuration in equilibrium is called virial theorem. Our relation $E_T \sim E_G$ constitutes a specific case of the virial theorem. The total energy $E = E_G + E_T$ scales in the same way:

$$-E \sim \frac{GM^2}{R} \sim \frac{M}{\mu} kT \tag{2.22}$$

For the star to remain bound, the total energy $-E > 0$. Otherwise expansion of a star into a cloud of infinite size $(R \to \infty)$ could occur spontaneously as an energy decreasing process.

2.3.3 Time scales

It is important to estimate characteristic time scales for restoration of balance in a star. The dynamical imbalance occurs when gas pressure P_g fails to balance gravity. Let us consider an extreme case of $P_g = 0$. The response of gas to such an imbalance is essentially free fall. For rough estimate of the dynamical time scale τ_D we take free fall time over distance of stellar radius R. Approximating gravity acceleration by a fixed average value $g \sim GM/R^2$ we obtain $R \sim g\tau_D^2/2$. By solving we obtain the following dynamical time scale:

$$\tau_D \sim \sqrt{\frac{R^3}{GM}} \sim \frac{1}{\sqrt{G < \rho >}} \sim 1 hour \tag{2.23}$$

The final number corresponds to the solar value. Restoration of hydrostatic equilibrium in a star occurs on the dynamical time scale, very short in cosmic terms. Note that the dynamical time scale is roughly matched by the period of the Kepler orbit grazing stellar surface. We shall demonstrate that pulsations of stars occur on a similar time scale.

The thermal or Kelvin-Helmholtz time scale τ_T yields estimate of cooling time of a star devoid of energy sources. For rough estimate we assume fixed luminosity

L during cooling so that $L\tau_T \sim E_T$. Substituting Eq. (2.22) one obtains

$$\tau_T \sim \frac{GM^2}{RL} \sim 10^7 years \tag{2.24}$$

Restoration of thermal (energy) equilibrium by cooling or heating of a star occurs by the thermal time scale.

The thermal time scale is short compared to the geological age of the Solar System estimated at $4 \cdot 10^4 y$. To maintain its near constant luminosity over so long time our Sun requires powerful source of energy. The source of energy turns out to be thermonuclear reactions. In these reactions 4 hydrogen nuclei are converted to a helium atom. However mass of 4 protons exceeds mass of the resulting helium atom by a fraction

$$\frac{4\mu_H - \mu_{He^4}}{\mu_{He^4}} \approx 0.007 \tag{2.25}$$

Corresponding fraction of mass is lost in these reactions and is converted to energy at the exchange rate established by Einstein at c^2. The time of burning of the whole hydrogen is called nuclear time scale τ_N. To simplify our estimate we assume that our star consists of pure hydrogen and burns it all. By equating losses with supplies we obtain: $L\tau_N \sim 0.007Mc^2$ hence

$$\tau_N \sim \frac{0.007Mc^2}{L} \sim 10^{10} years \tag{2.26}$$

The nuclear time scale exceeds the age of the Solar System by comfortable margin and is close to the age of the whole universe.

Note that the dynamical time scale is by so many orders of magnitude short than the thermal time scale that during restoration of dynamical equilibrium occurs negligible heat flow. In other words the corresponding process is adiabatic. On the contrary restoration of the energy (thermal) equilibrium occurs at such a slow pace that the star has time to maintain hydrostatic equilibrium all time.

2.4 Stability of hydrostatic equilibrium

2.4.1 Dynamical stability criterion

To check stability of mechanical equilibrium of a star we study evolution of small perturbations of stellar size R. Let us consider response of hydrostatic (gravity) and gas pressures to the corresponding perturbations of density $\rho \sim R^{-3}$. Restoration of hydrostatic equilibrium occurs on dynamical time scale. The dynamical time scale is so short compared to the thermal time scale that negligible heat flow occurs in a star before restoration of the equilibrium. Thus for these fast processes holds adiabatic equation of state $P_g \sim \rho^\gamma$ (Eq. 2.14). Formally the hydrostatic pressure at center may be considered polytropic function of density too (Eqs. 2.18). Hence their ratio also scales with power of density:

$$\frac{P_c}{P_g} \sim \frac{G}{K} M^{2/3} \rho^{4/3-\gamma} \tag{2.27}$$

The hydrostatic pressure acts inwards, reflecting gravity compression of a star. The gas pressure acts outwards. For a star originally remaining in mechanical equilibrium these pressures have to balance $P_c/P_g = 1$. Let us consider a perturbation decreasing size and increasing average density ρ. For stable equilibrium the perturbation should produce excess gas pressure $P_c/P_g < 1$ to drive expansion to the original size. Inspection of the exponent of ρ in Eq. (2.27) reveals that this occurs if and only if

$$\gamma > \frac{4}{3} \tag{2.28}$$

The reader may check that same conclusion follows for initial expansion of a star. Thus Eq. (2.28) yields criterion for dynamical stability of a star.

2.4.2 Radial Pulsations

Let us consider a simplistic model of stellar pulsation: oscillation of an envelope of mass $\boxed{M}/2$ is driven by imbalance of hydrostatic and gas pressures of order half of the central pressure, $\boxed{P_h}/2$ $\boxed{P_g}/2$, at a surface expanded by δR from its equilibrium radius of $\boxed{R}/2$. Taking only framed values and neglecting factors of order 1 we obtain the following equation of motion:

$$-\frac{d^2\delta R}{dt^2} \sim \frac{R^2 P}{M} \frac{\partial(\ln P_h - \ln P_g)}{\partial\rho} \delta rho \sim \frac{GM}{R^3}(3\gamma - 4)\delta R \tag{2.29}$$

where we substituted $\delta P = P\delta \ln P$ and $\delta\rho \sim \rho\delta \ln(M/R^3) \sim -3(M/R^3)\delta R/R$. The resulting equation is equation of harmonic oscillations. Substitution of a solution in the form $\delta r \sim e^{i\omega t}$ yields the stability criterion

$$\omega^2 = \frac{GM}{R^3}(3\gamma - 4) > 0 \tag{2.30}$$

i.e. Eq. (2.28). For $\omega^2 < 0$ solution becomes exponentially instable $\delta R \sim e^{\pm|\omega|t}$.

Chapter 3

Polytropic stars

For a special case of barotropic equation of state, namely for polytropic equation of state the stellar structure equations admit exact analytical solutions. By no means these solutions are of purely academic interest. Some of them approximate well structure of existing stars, such as white dwarfs and, to a lesser degree, neutron stars. The price we pay for it are longer then elsewhere in this course analytic expressions although the involved mathematics is no more difficult then elsewhere.

3.1 Virial theorem and stability

3.1.1 Evaluation of energies

Let us start from consideration of an auxiliary integral

$$I = \int P dV = \int_0^R P 4\pi r^2 dr > 0 \tag{3.1}$$

The integral value is positive for the integrated function is everywhere positive. Now we proceed to evaluation of the gravity potential energy. In doing so we add contributions coming from consecutive shells of mass dm. To transform the expression for gravity energy we substitute the continuity equation (Eq. 2.1) for dm, the hydrostatic equilibrium equation for dP/dr (Eq. 2.4) and finally integrate by parts:

$$\begin{aligned} E_G &= -\int_0^M \frac{Gmdm}{r} = -\int_0^M \frac{Gm}{r} 4\pi\rho r^2 dr = -\int_0^M \frac{Gm\rho}{r^2} 4\pi r^3 dr = \\ &= \int_0^M \frac{dP}{dr} 4\pi\rho r^3 dr = P 4\pi r^3\Big|_0^R - 3\int_0^R P 4\pi r^2 dr = -3I \end{aligned} \tag{3.2}$$

First term vanishes by virtue of the boundary conditions.

The thermal energy may be calculated by summing of the gas energy U over the whole star:

$$E_T = \int U dm \tag{3.3}$$

Evaluation of the integral involves a trick. Note that U depends solely on the state of gas and not on the way this state was achieved. So for computational purposes we are free to assume that the star was build by an adiabatic contraction in which everywhere entropy remained constant $\delta S = 0$. Care must be exercised in notation to differentiate between differentials due to contraction and due to integration over volume. The former are indicated by symbol δ, the latter by d, respectively. We employ first thermodynamic principle in the form $\delta U = T\delta S - P\delta(1/\rho)$. First term on the R.H.S. vanishes for adiabatic contraction, so we obtain:

$$\delta E_T = \int \delta U dm = -\int P\delta(1/\rho) dm = \int K\rho^{\gamma-2}\delta\rho dm \tag{3.4}$$

Now we integrate over the δ symbols. Physically this means adding infinitesimal changes occurring due to contraction of the initial gas cloud. We start contraction from the cloud of infinite size of near zero temperature $(E_T = 0)$ and follow it till we reach the actual state of the star. Because of mass conservation this integration does not affect integration over stellar mass dm. In this way we obtain

$$E_T = \frac{K}{\gamma - 1}\int \rho^{\gamma-1} dm = \frac{1}{\gamma - 1}\int \frac{P}{\rho} dm = \frac{I}{\gamma - 1} \tag{3.5}$$

3.1.2 The virial theorem

By elimination of I from Eqs. (3.2) and (3.5) we obtain the virial theorem for polytropic stars

$$3(\gamma - 1)E_T = -E_G \tag{3.6}$$

The global energy of a polytropic star is

$$E = E_T + E_G = -\frac{3\gamma - 4}{\gamma - 1} I \tag{3.7}$$

For dynamical stability of configuration against expansion into infinity the global energy must be negative. For I positive by construction this means either $\gamma < 1$ or $3\gamma > 4$. The former condition is consistent with no physical system, since it means cooling by adiabatic compression (c.f. $\gamma = 1$ for isothermal compression). In this way we obtain by formal analytical calculation the stability criterion for the dynamical equilibrium of a polytropic star

$$\gamma > \frac{4}{3} \tag{3.8}$$

Note curios property of a star: its global 'specific heat' is negative. We mean by this that adding energy to a star decreases its internal temperature. To demonstrate as much let us point that $I = -(1/3)E_G \sim +GM^2/R$ (Eq. 2.20). Adding energy to a star increases E. Since it E is negative it means that for $\gamma > 4/3$ I decreases so E_T decreases too. Since E_T is summed U and for a perfect gas $U = c_V T$ where the gas specific heat c_V is constant, the temperature T decreases.

3.2 Polytropic stars

3.2.1 Lane-Emden equation

The stellar structure equations relevant for a polytropic star are Eqs. (2.1), (2.4) and (2.14):

$$dm = 4\pi r^2 \rho dr \quad dP = -\tfrac{Gm}{r^2}\rho dr \quad P = K\rho^{1+1/n}$$

In these equations we replace ρ and r by dimension-less variables θ and ξ defined in the following way:

$$\begin{aligned} \rho &= \rho_c \theta^n \\ r &= r_n \xi \end{aligned} \tag{3.9}$$

where ρ_c and r_n are constants meaning central density and yet unspecified unit of length. Next we solve the hydrostatic equilibrium equation for m, differentiate over ξ and substitute into the continuity equation. In this way we obtain Lane-Emden second order differential equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \quad \text{for } r_n \text{ satisfying} \tag{3.10}$$

$$\left(\frac{P_c(n+1)}{4\pi G\rho_c^2}\right)^{1/2} = r_n. \tag{3.11}$$

In this way r_n no longer remains arbitrary. There are two boundary conditions at the center:

$$\theta(0) = 1 \qquad \frac{d\theta}{d\xi}(0) = 0 \tag{3.12}$$

The latter condition is consistent with vanishing of R.H.S. of the hydrostatic equation, $Gm\rho/r \to G\rho^2 4\pi r^3/3r \to 0$ for $r \to 0$. In this way for a given n one obtains unique solution of Lane-Emden equation. Physically valid solutions satisfy the surface condition:

$$\theta(\xi_1) = 0 \tag{3.13}$$

Table 3.1: Analytical solutions for polytropic stars

n	$\theta^{(n)}(\xi)$
0	$1 - \frac{\xi^2}{6}$
1	$\frac{\sin\xi}{\xi}$
5	$\frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$

Valid solutions exist for $0 < n < 5$. Negative n is unphysical for it means decrease of pressure with increasing density. Solutions for $n > 5$ for growing ξ first fall to a positive minimum value and than grow to infinity. They never satisfy Eq. (3.14).

3.2.2 Solutions

Several special values of n admit analytical solutions. They are listed in Table 3.1.

Solutions of Lane-Emden equation define a family of special functions called Lane-Emden functions. For symmetry reasons all odd derivatives of θ vanish for $\xi = 0$. Direct substitution into Lane-Emden equation demonstrates that $\theta^{(0)} = 1 - \xi^2/6$ is a valid solution for arbitrary n and small ξ, up to the order of ξ^3.

Exercise 3.2.1 (Numerical Solutions of Lane-Emden Equations) *For practical evaluation of solutions more useful is numerical integration of the corresponding system of first order equations:*

$$\frac{d\psi}{d\xi} = -\theta^n \xi^2 \tag{3.14}$$

$$\frac{d\theta}{d\xi} = -\frac{\psi}{\xi^2} \tag{3.15}$$

from the starting values

$$\psi(\epsilon) = -\frac{\epsilon^3}{3} \tag{3.16}$$

$$\theta(\epsilon) = 1 - \frac{\epsilon^2}{6} \tag{3.17}$$

Use any simple integration routine (e.g. Runge-Kutta) and try $\epsilon = 10^{-2}, 10^{-4}$ *and* 10^{-8}. *For* $n = 0, 1$ *and* 5 *perform integrations from* $\xi = \epsilon$ *till* $\xi = 1$. *Check for which* ϵ *correspondence of analytical and numerical solutions is best. For the best value of* ϵ *continue integrations till* $\theta = 0$ *and find corresponding* θ_1. *List* θ_1 *for* n *in the range* 0 – 4.5 *with step 0.5.*

3.2.3 Stellar parameters

Let us proceed to derivation of parameters of stars corresponding to Lane-Emden solutions. Using Eq. (2.1) and (3.10) we obtain

$$\begin{aligned} M &= \int \frac{dm}{dr} dr = 4\pi r_n^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi = 4\pi r_n^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = \\ &= -4\pi \left(\frac{K(n+1)}{4\pi G}\right)^{3/2} \rho_c^{\frac{3-n}{2n}} \left(-\xi^2 \frac{d\theta}{d\xi}\right)\Bigg|_{\xi_1} \end{aligned} \tag{3.18}$$

Similarly

$$R = r_n \xi_1 = \left(\frac{K(n+1)}{4\pi G}\right)^{1/2} \rho_c^{\frac{1-n}{2n}} \xi_1 \tag{3.19}$$

Eliminating ρ_c from both equations we obtain

$$M^{\frac{n-1}{n}} R^{\frac{3-n}{n}} = \frac{K(n+1)}{G(4\pi)^{1/n}} \left[\left(-\xi^{\frac{n+1}{n-1}} \frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{\frac{n-1}{n}} = const \tag{3.20}$$

In this way we obtained the mass-radius relation for polytropic stars. The relation may be verified by observations of degenerate stars and has many applications.

Other useful parameter combinations are

$$\frac{\rho_c}{<\rho>} = \frac{4\pi\rho_c R^3}{3M} = -\frac{\xi_1}{3 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}} \tag{3.21}$$

$$\frac{P_c}{\rho_c}(n+1)(-\xi\frac{d\theta}{d\xi})_{\xi_1} = \frac{GM}{R} \tag{3.22}$$

Surface parameters of polytropic stars are listed in Table 3.2.

3.2.4 Evaluation of the energy integral

Evaluation of the energy integral (Eq. 3.1) is a little bit involved. First we turn to dimension-less variables:

$$I = \int P dV = 4\pi P_c r_n^3 \int_0^{\xi_1} \theta^{n+1} \xi^2 d\xi \tag{3.23}$$

We proceed to setting of an algebraic equation for the unknown integral. For its L.H.S. we substitute Lane-Emden equation into dimension-less integral and integrate by parts. For simplicity of notation we omit integration boundaries:

$$\int \theta^{n+1}\xi^2 d\xi = -\int \theta \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = 0 + \underline{\int \xi_2 \left(\frac{d\theta}{d\xi}\right)^2 d\xi} \tag{3.24}$$

Table 3.2: Surface values for polytropic stars

n	$\xi_1^{(n)}$	$-\xi^2 \frac{d\theta}{d\xi}\Big\vert_1$	$\frac{\rho_c}{<\rho>}$
0	2.4494	4.8988	1.0000
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
3.0	6.89685	2.01824	54.1825
4.0	14.97155	1.79723	622.408
4.5	31.8365	1.73780	6189.47
5.0	∞	1.73205	∞

(from Chandrasekhar, 1939)

For the R.H.S. we start from the same integral but integrate by parts first and only later substitute Lane-Emden equation:

$$
\begin{aligned}
\int \theta^{n+1}\xi^2 d\xi &= \frac{1}{3}\int \frac{d}{d\xi}\left(\xi^3\right)\Theta^{n+1} d\xi = 0 - \frac{n+1}{3}\int \xi^3 \left(\frac{d\theta}{d\xi}\right)^2 d\xi = \\
&= \frac{n+1}{3}\int \xi \frac{d\theta}{d\xi}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = \frac{n+1}{6}\int \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)^2 d\xi = \\
&= \frac{n+1}{6}\left\{\left(\xi^2 \frac{d\theta}{d\xi}\right)^2 \frac{1}{\xi}\Bigg|_0^{\xi_1} + \int \left(\xi^2 \frac{d\theta}{d\xi}\right)^2 \frac{d\xi}{\xi^2}\right\} = \\
&= \frac{n+1}{6}\left\{\xi^3 \left(\frac{d\theta}{d\xi}\right)^2 + \underline{\int \xi^2 \left(\frac{d\theta}{d\xi}\right)^2 d\xi}\right\}
\end{aligned}
$$

By equating two sides and solving for the underlined term we obtain

$$
\frac{n+1}{5-n}\,\xi^3 \left(\frac{d\theta}{d\xi}\right)^2\Bigg|_{\xi_1} = \int \xi_2 \left(\frac{d\theta}{d\xi}\right)^2 d\xi = \int \theta^{n+1}\xi^2 d\xi \tag{3.25}
$$

where the last equality is copied from the L.H.S. Substitution of this result into Eqs. (3.23) (3.2), (3.5) and (3.8) yields the final forms for I and energies of a polytropic star

$$
I = \frac{P_c}{\rho_c} M \frac{n+1}{5-n}\left(-\xi \frac{d\theta}{d\xi}\right)_{\xi_1} = \frac{1}{5-n}\frac{GM^2}{R} \tag{3.26}
$$

$$
E_G = -\frac{3}{5-n}\frac{GM^2}{R} \tag{3.27}
$$

$$E_T = \frac{n}{5-n}\frac{GM^2}{R} \tag{3.28}$$

$$E = \frac{n-3}{5-n}\frac{GM^2}{R} \tag{3.29}$$

Chapter 4

Energy equilibrium

For non-barotropic equations of state, i.e. for those which depend on temperature, mechanical and energy equilibria are coupled. To find stellar structure we have to consider production and transport of energy. So the final set of differential equations of stellar structure consists of two equations derived from the mechanical equilibrium and two equations derived from the energy equilibrium.

4.1 Energy production equation

Energy conservation may be considered in a similar way to mass conservation in the continuity equation (Eq. 2.1). The rate of energy production (power) per unit mass of gas is a new state function $\epsilon = \epsilon(\rho, T, \cdots)$. Let us define a new variable l as total power of energy sources contained within radius r of the center of a star. Then energy production in a thin spherical shell is

$$dl = 4\pi r^2 \rho\epsilon dr \tag{4.1}$$

This equation is called the energy production equation. The corresponding boundary condition is

$$l(0) = 0 \tag{4.2}$$

The luminosity L is defined in terms of l in the following way

$$L = l(R) \tag{4.3}$$

Several physical processes contribute to the energy production rate ϵ:

nuclear reactions (energy source) Temperatures in stellar interiors are high enough for occurrence of thermonuclear reactions. In stellar conditions fusion of light nuclei is irreversible and produces copious amounts of energy for generally the mass of product nucleus is ba bit less than the mass of its components.

Table 4.1: Processes transporting energy in stars

process name	energy form	carrier	carrier energy	carrier range
radiation	total energy	photon	$< 1keV$	$1cm$
conduction	kinetic energy	electrons	$< 1keV$	
convection	heat	gas eddies		10^8cm

neutrino emission (energy sink) Some nuclei produced in stellar interiors are unstable against β decay, i.e. emission of an electrons. These decays are sometimes followed by a reverse process, namely by electron capture, so that no change of composition arises. This process, called 'Urca' occurs at very high densities, when crowded free electrons occupy all states up to energies comparable to the β decay energy. However, in both β decays and electron capture are emitted very light but energetic particles, neutrinos. Their interaction with the stellar matter is so weak that most of them leaves the star, carrying away energy.

work by gravity (energy source or sink) During evolution stars may reach stages when mechanical and thermal equilibria no longer could be maintained. Resulting changes of their structure, contraction or expansion involve gas motions in which work is done or lost by the gravity field.

4.2 Energy transport

There are three processes capable of transporting energy in stars. They employ different carriers and forms of energy, as listed in Table 4.1. The number listed in the table correspond to deep interior conditions. Near-surface values are different.

General equation of energy transport may be written in the form familiar from the heat conduction:

$$F = D\frac{dT}{dr} \tag{4.4}$$

With F and D we denote energy flux (power per unit surface) and conduction coefficient, respectively. We shall demonstrate that this form holds as well for radiation and convection, with appropriate definitions of D.

4.3 Transport by radiation

Energy transport by radiation is discussed at length in the course of stellar atmospheres. In stellar interiors applies simple approximation called the diffusion approximation. It relies on the assumption that within the range of photons only very slight deviations occur from the thermodynamic equilibrium. In particular radiation is near isotropic and homogeneous. For completeness of the course we rederive here main results of the diffusion approximation.

Let n and σ denote respectively density of particles and cross section for their interaction. The cross section is defined in such a way that the probability of interaction of a particle with a flux of photons is the same as the probability of these photons to hit a target of surface σ. The combined cross section of all particles in a unit of volume or mass, $K = n\sigma$ or $\kappa = n\sigma/\rho$ respectively, is called opacity. Mean free path Λ of a photon is length of gas column in which sum of cross section is capable to cover the column cross section. In other words, travelling distance of about Λ a photon is likely to interact wit at least one particle. For unit cross section of the column we obtain:

$$1 = K\Lambda = \kappa\rho\Lambda \tag{4.5}$$

In the Sun a typical value of density and opacity is $\rho \sim 1g/cm^3$ and $\kappa \sim 1cm^2/g$ hence the mean free path is

$$\Lambda = \frac{1}{\kappa\rho} \sim 1cm \tag{4.6}$$

Astrophysicists prefer opacity κ to individual cross sections σ, so in this course we employ σ to denote Stefan-Boltzmann constant, unless explicitly stated otherwise.

The radiation field may be described by its intensity I and net flux F. Intensity I is energy per unit time (power) carried by photons passing unit surface whose directions fall within a unit spherical angle. Mean free path is so much less than stellar radius that we are able to approximate a shell crossed by a photon with a plane slab of gas. In such a geometry intensity of radiation is a function of wavelength λ, radius (location) r and cosine of the angle between photon direction and normal to the plane surface μ: $I = I_\lambda(\mu, r)$. Let us expand I in a Taylor series of μ:

$$I_\lambda = I_\lambda^{(0)} + \mu I_\lambda^{(1)} + \cdots \tag{4.7}$$

In this expansion $I_\lambda^{(1)}$ is responsible for asymmetry of radiation. The asymmetry is caused by photons arriving from other shells at average distance Λ and temperature difference ΔT. Thus for any smooth dependence of emitted I on temperature T we have $I_\lambda^{(1)}/I_\lambda^{(0)} \sim \Delta T/T \sim \Lambda/R \sim 10^{-10}$. The radiation field inside of stars is indeed very closely isotropic and in balance with nearby gas with close temperature, hence its intensity has near thermodynamic equilibrium distribution $I_\lambda^{(0)} \approx B_\lambda$, i.e. black body distribution. The net flux F_λ is intensity from a projected surface integrated over the full spherical angle Ω. Projection of surface introduces an additional factor of μ:

$$F_\lambda = \int I_\lambda \mu d\Omega = 2\pi \int_{-1}^{+1} I_\lambda^{(1)} \mu^2 d\mu = 4\pi I_\lambda^{(1)}/3. \tag{4.8}$$

Note that because of antisymmetry terms with odd powers of μ vanish. In this way first terms of Taylor expansion of intensity may be written as

$$I_\lambda = B_\lambda + \mu \frac{3}{4\pi} F_\lambda + \cdots \tag{4.9}$$

Integral (bolometric) flux from unit surface of a black body by virtue of Stefan-Boltzmann law is σT^4. This flux may be expressed in terms of bolometric intensity B (Eq. 2.10):

$$\sigma T^4 = F = \pi B \tag{4.10}$$

Let us consider oblique transfer of radiation through a plane parallel slab of gas of thickness dr. The crossing length is dr/μ. Intensity of the crossing radiation changes by dI_λ due to absorption and emission. Absorption is proportional to the product of crossing length, opacity and intensity $(dr/\mu)K_\lambda I_\lambda$. By recalling definition of K_λ note in this respect that $(dr/\mu)K_\lambda$ corresponds to the fraction of surface of slab covered by particle cross sections. Emission X_λ occurs in random directions, i.e. it is isotropic. In this way we obtain the radiation transfer equation

$$dI_\lambda = (X_\lambda - K_\lambda I_\lambda) dr/\mu. \tag{4.11}$$

Let us multiply the transfer equation by μ^2, substitute Eq. (4.9) and integrate over the full spherical angle, remembering that contribution of odd powers of μ vanishes (c.f. Eq. 4.8). In this way we obtain

$$\frac{4\pi}{3} dB_\lambda = (0 + K_\lambda F_\lambda) dr \tag{4.12}$$

Next we divide result by K_λ and integrate over λ

$$\frac{4\pi}{3\tilde{K}} dB = F dr \tag{4.13}$$

Symbol $\tilde{K}$ indicates so called Rosseland mean opacity defined as

$$\frac{1}{\tilde{K}} \frac{dB}{dT} \equiv \int \frac{1}{K_\lambda} \frac{dB_\lambda}{dT} d\lambda \tag{4.14}$$

Changing variables of differentiation and substituting Eq. (4.10) we obtain $\pi dB = (dB/dT)dT = 4\sigma T^3 dT$. Next we express luminosity using flux: $l = 4\pi r^2 F$. Substitution of these relation into Eq. (4.14) yields

$$\frac{16\sigma T^3}{3\tilde{K}} \frac{dT}{dr} = F = \frac{l}{4\pi r^2} \tag{4.15}$$

Solving for T/dr we obtain the customary form of the radiative transport equation:

$$dT = -\frac{l}{4\pi r^2} \frac{3}{16} \frac{\tilde{K}\rho}{\sigma T^3} dr \tag{4.16}$$

Comparison with Eq. (4.6) reveals form of the radiative transport coefficient $D = (16/3)(\sigma T^3/\tilde{K}\rho)$.

4.4 Conduction

Gases in ordinary conditions are poor heat conductors. So conduction plays no role in ordinary stars of solar type. Conduction by free electrons is possible only when most accessible electronic states are occupied, as happens in the degenerate gas. In such conditions electron travel long distances Λ before finding an unoccupied state to scatter into it. Large Λ indicates small equivalent 'opacity' K_c and effective heat transport (c.f. Eq. 4.16). When two mechanisms of energy transport with opacities K_r and K_c operate in parallel, we ought to add corresponding fluxes. As $F \sim 1/K$, we find the effective opacity K from the following equation:

$$\frac{1}{K} = \frac{1}{K_r} + \frac{1}{K_c} \tag{4.17}$$

Note analogy with parallel connection of resistors.

4.5 Convection

4.5.1 Acoustic time-scale

By definition of the absolute temperature scale the average kinetic energy of a particle of gas $\mu v_T^2 \sim kT$. The thermal and sound velocities v_T and v_S agree within a factor of order 1. In the same approximation from the perfect gas equation (Eq. 2.6) one obtains $P/\rho \sim kT/\mu \sim \mu v_T^2/\mu \sim v_S^2$ or finally

$$v_S \sim \frac{P}{\rho} \tag{4.18}$$

It is convenient to introduce here a pressure scale-height H defined as

$$H = \frac{1}{\frac{-d\ln P}{dr}} = \frac{P}{-\frac{dP}{dr}} \sim \frac{P}{g\rho} \tag{4.19}$$

Last approximation based on Eq. (2.4) holds for smooth pressure distribution in stellar interior but breaks at the surface. Let us introduce the acoustic frequency as frequency of the sound waves bouncing over distance H and transform it using Eqs. (4.18) and (4.19):

$$\omega_A^2 \sim \frac{v_S^2}{H^2} \sim \frac{P}{\rho}\frac{g\rho}{HP} \sim \frac{g}{H} \tag{4.20}$$

This frequency also related to the frequency of dynamical oscillations of wavelength H and inverse of time of restoration of hydrostatic equilibrium in layer of thickness H. Sound waves involve alternate expansion and compression of gas due to action of the alternate pressure excess and deficiency.

4.5.2 Stability against convection

Convection is driven by buoyancy force and not by pressure (compression). So are gravity waves, such as waves on an ocean. Hence it is illuminating to consider convection as a special case of gravity waves in stars. Let us consider evolution of a bubble of gas moving vertically in stellar interior over distance δr. Initial conditions at $\delta r = 0$ inside and outside of the bubble were the same. The final conditions inside and outside may differ, so we indicate respective changes of density and pressure in the new location with $\delta\rho_i$, δP_i and $\delta\rho_o$, δP_o, respectively. Here δ indicates difference between initial and final conditions. With $\Delta\rho = \delta\rho_i - \delta\rho_o$ we indicate difference between final density inside and outside. We assume that motion is much slower than sound velocity, hence hydrostatic equilibrium is preserved

$$\delta P_i = \delta P_o. \tag{4.21}$$

By differentiation of the equation of state in the form $P = P(\rho, T)$ the hydrostatic equilibrium may be written as $0 = \Delta P = (\partial P/\partial\rho)_T \Delta\rho + (\partial P/\partial T)_\rho \Delta T$. For the perfect gas equation of state (Eq. 2.6) $\delta \ln P = \delta \ln \rho + \delta \ln T$, hence $1 = (\partial \ln P/\partial \ln \rho)_T = (\rho/P)(\partial P/\partial\rho)_T$ and similarly for the other derivative. Substituting we obtain for a buble in hydrostatic equilibrium

$$\frac{\Delta\rho}{\rho} = -\frac{\Delta T}{T} \tag{4.22}$$

where as before symbol Δ indicates of difference between conditions inside and outside of bubble.

The force of buoyancy depends on $\Delta\rho$. The equation of motion of unit volume of gas driven by buoyancy is

$$\begin{aligned} -\rho\frac{d^2\delta r}{dt^2} &= g\Delta\rho = g\rho\frac{d\Delta\ln\rho}{d\ln P}\frac{d\ln P}{dr}\delta r = \\ &= \frac{g\rho}{H}B\delta r = \omega_A^2\rho B\delta r \quad \text{where} \end{aligned} \tag{4.23}$$

$$B = -\frac{d\Delta\ln\rho}{d\ln P} \tag{4.24}$$

In transformations we exploited Eqs. (4.19) and (4.21). The latter equation enabled adoption of pressure as a general variable of differentiation in expression for B. The result is a harmonic oscillator equation.

Substituting vertical shift $\delta r \sim e^{i\omega t}$ into Eq. (4.23) and transforming we obtain frequency of gravity oscillations

$$\omega_{B-V}^2 = \omega_A^2 B \tag{4.25}$$

This frequency is called Brunt-Väisäla frequency. Further motion of the bubble depends on sign of B. For $B > 0$ an initial push results in gravity oscillations. It also means no convection. For $B < 0$ we have an instability, i.e. convection. Note that sign of B depends on sign of $\Delta\rho$ for an initial push of the bubble outwards, i.e. towards smaller pressure $\delta P < 0$. For $\Delta\rho > 0$ bubble sinks, i.e. returns to equilibrium. This means stability against convection.

The criterion $B > 0$ may be transformed to its more customary form exploiting the equation of state $\rho = \rho(P, T, \mu)$. Differentiating it with respect of ΔT and $\Delta\mu$ and substituting into Eq. (4.24) we obtain

$$-B = \frac{\partial \ln \rho}{\partial \ln T}\left(\frac{d \ln T}{d \ln P_i} - \frac{d \ln T}{d \ln P_o}\right) + \frac{\partial \ln \rho}{\partial \mu}\left(0 - \frac{d \ln \mu}{d \ln P_o}\right) < 0. \tag{4.26}$$

Note that ΔP vanishes by virtue of Eq. (4.21) and chemical composition of the bubble remains intact $\delta\mu_i = 0$. However, we do take into account variation of the chemical composition with radius $\delta\mu_o$ present in many evolved stars. For $P, \mu = const$ in a non-pathological equation of state $(-\partial \ln \rho / \partial \ln T) > 0$. Dividing Eq. (4.26) by this expression we obtain the convection criterion in the form first derived by Ledoux:

$$\begin{aligned}
\nabla_o &< \nabla_i + \frac{\frac{\partial \ln \rho}{\partial \ln \mu}}{-\frac{\partial \ln \rho}{\partial \ln T}}\nabla_\mu \quad \text{where} & (4.27)\\
\nabla_o &\equiv \frac{d \ln T}{d \ln P_o} \\
\nabla_i &\equiv \frac{d \ln T}{d \ln P_i} \\
\nabla_\mu &\equiv \frac{d \ln \mu}{d \ln P_o} \quad \text{and} \\
B &= \left(-\frac{\partial \ln \rho}{\partial \ln T}\right)\left(\nabla_i - \nabla_o + \frac{\frac{\partial \ln \rho}{\partial \ln \mu}}{-\frac{\partial \ln \rho}{\partial \ln T}}\nabla_\mu\right) & (4.28)
\end{aligned}$$

Note that for mean particle mass growing inwards $\nabla_\mu > 0$ the Ledoux inequality becomes sharper, hence star becomes more stable against convection. This has an effect in red giant stars at the interface of the hydrogen rich envelope with convection and helium nucleus without convection.

The Ledoux criterion may be simplified for homogenous gas $\nabla_\mu = 0$ deep in stellar interiors where matter is dense and so opaque, that bubbles loose very little energy by radiation and follow adiabatic process $\nabla_i \approx \nabla_{ad}$:

$$\nabla_{rad} < \nabla_{ad} \tag{4.29}$$

As further simplification in Eq. (4.29) we replaced ∇_o with its value for pure radiative transport $\nabla_{rad} = (P/T)(dT/dr)/(dP/dr)$ derived from Eqs. (4.16) and (2.4). This criterion is called Schwarzschild criterion.

In a general case when transport occurs both by radiation and convection the following inequalities hold:

$$\nabla_{rad} > \nabla_o > \nabla_i > \nabla_{ad} \tag{4.30}$$

The first inequality sign indicates that radiation carries only part of energy, the second is consistent with presence of convection and third indicates presence of some radiative losses from bubbles. For adiabatic process of a perfect gas $\delta \ln P = \gamma\delta\rho$ and $\delta \ln P = \delta \ln \rho + \delta \ln T$ (Eqs. 2.14 and 2.6). Elimination of $\delta \ln \rho$ yields

$$\nabla_{ad} = \frac{\gamma - 1}{\gamma} \approx \frac{2}{5} \tag{4.31}$$

The last value corresponds to the monoatomic gas with $c_P = 3k/2\mu$ and $c_V = 5k/2\mu$.

4.5.3 Convective equation of transport

For convection in deep stellar layers ΔT of order of several degrees is negligible compared to $T \sim 10^7 K$. This will be demonstrated in Sect. 4.6. Hence in stellar interior we may assume that in the convective layers $\nabla_o \approx \nabla_{ad}$ and

$$\frac{dT}{dr} = \frac{T}{P}\nabla_{ad}\frac{dP}{dr} \tag{4.32}$$

The R.H.S. is a known function of parameters. In particular its last factor is known from Eq. (2.4). For convective layers deep in stellar interior convection becomes (near) adiabatic and Eq. (4.32) replaces the radiative transfer equation. For subsurface layers convection becomes non-adiabatic and one has to recall the mixing length theory (Sect. 4.6).

Convective transport operates in stars in two basic situations: either the radiative gradient becomes large $\nabla_{rad} \nearrow$ or the adiabatic gradient becomes small $\nabla_{ad} \searrow$, as listed in Table 4.2. By the thermostatic effect we understand an effect resembling phase transition, e.g. melting of ice. Heating of a mixture of ice and water does not change its temperature but alters composition by melting more ice. Analogy is not full, as during partial ionization temperature does not remain fixed but raises slowly with heating, resulting in ∇_{ad} positive small as much energy is consumed by progressive ionization of gas.

Table 4.2: Causes of convection

Responsible parameter	Scenario
$\kappa \nearrow \Rightarrow \nabla_{rad} \nearrow$	partial ionization of envelopes of cool stars
$F \nearrow \Rightarrow \nabla_{rad} \nearrow$	cores of blue giants (massive main sequence stars)
$\nabla_{ad} \searrow$	thermostatic effect of partial ionization layers in cool stars

4.5.4 Side-effects of convection

Overshooting

Convective elements have inertia, so their breaking length in layers stable against convection is finite. In this way they can penetrate certain distance into those layers. This affects layers of strong chemical gradient which are stable against convection. Overshooting from above tends to mix slowly convective envelope with stable layer from below. Another effect of overshooting is solar granulation. Top layers of Sun are stable against convection. However, overshooting from the convective zone in the envelope below produces characteristic pattern on the surface.

Mixing of chemical composition

Mixing of convective elements occurs on time of order $1/\omega_{B-V}$, short compared to the thermal and nuclear time scales. Hence in presence of convection chemical composition in the whole convective zone of mass Δm is uniform, amounting to $< X >$. Evolution of the composition occurs due to burning of matter inside of the convection zone at the rate $\partial X/\partial t$ and by mixing of external material at the rate $\partial m/\partial t$:

$$\frac{\partial < X >}{\partial t} = \frac{1}{\Delta m}\left(\int_{\Delta m} \frac{\partial X}{\partial t}\delta m + \frac{\partial m}{\partial t}(X - < X >)\Big|\right) \tag{4.33}$$

Convective mixing is responsible for destruction of lithium in the envelopes of low mass main sequence stars. Lithium is burned by fusion with hydrogen at fairly low temperatures:

$$\begin{array}{lll} Li^6 + p \rightarrow He^4 + He^3 & \text{for} & T > 2.0 \cdot 10^6 \\ Li^7 + p \rightarrow He^4 + He^4 & \text{for} & T > 2.4 \cdot 10^6 \end{array}$$

Young stars indicated by presence of circumstellar envelopes and excess infrared emission from surrounding dust contain an admixture of lithium from the interstellar matter in amounts spectroscopically detectable. Stars less massive than the Sun have extended convection zones in their envelopes. The convection does not reach central region burning hydrogen. However it does reach layers of at which lithium may burn. In this way in lithium is destroyed soon after formation of a star. Hence presence of lithium is signature of young age of a star. In ordinary unevolved main sequence stars lithium is already missing.

4.6 Mixing length theory

Let us face the truth: the mixing length theory is unpleasant in many ways. It is unpleasant for the student for it involves lengthy transformations. It is also unpleasant for the researcher, for it involves incomplete understanding of physics and a free parameter of a priori unknown value. Unfortunately it is the only theory of non-adiabatic convection we have.

4.6.1 Contribution by radiation

In a general case of non-adiabatic convection both $\nabla_i - \nabla_o$ and $\nabla_i - \nabla_{ad}$ have to be calculated from the theory. The free parameter is an average distance travelled by a bubble, i.e. the mixing length h. It is customary to express h in units of H: $\alpha = h/H$. The size of a bubble is approximated with H. To simplify derivation we assume $\nabla_\mu = 0$, as is usually the case in the envelope layers where the mixing length theory applies.

First equation reflects parallel transport of energy by convection and radiation:

$$\boxed{\frac{l}{4\pi r^2} = F_c + F_r} \equiv \frac{16}{3}\frac{\sigma T^3}{\tilde{K}P}\nabla_{rad}\frac{dP}{dr} \tag{4.34}$$

where the last equality defines ∇_{rad}. Second equation is plainly a copy of Eq. (4.16):

$$\boxed{F_r = \frac{16}{3}\frac{\sigma T^3}{\tilde{K}P}\nabla_o\frac{dP}{dr}} \tag{4.35}$$

4.6.2 Contribution by convection

Convective motion of a gas bubble may be approximated with gravity oscillation. A bubble travels distance h in time $\tau_{B-V} \equiv 1/\omega_{B-V}$. Its velocity expressed in terms of ω_{B-V} (Eqs. 4.25, 4.28) is:

$$\boxed{v^2 \sim} h^2\omega_{B-V}^2 \sim \boxed{\alpha^2 Hg\left(-\frac{d\ln\rho}{d\ln T}\right)(\nabla_o - \nabla_i)} \tag{4.36}$$

The temperature excess $\Delta \ln T$ in terms of h is: $\Delta \ln T = (\nabla_o - \nabla_i)(d \ln P/dr)h = (\nabla_o - \nabla_i)\alpha$ (c.f. Eq. 4.19). Now the convective flux F_c may be calculated as excess heat carried by bubbles:

$$\boxed{F_c \sim} vC_P T \Delta \ln T \sim \boxed{vC_P T(\nabla_o - \nabla_i)\alpha} \tag{4.37}$$

For an estimate of $\nabla_o - \nabla_i$ it is convenient to square Eq. (4.37) and to substitute into it Eq. (4.36):

$$F_c^2 \sim \alpha^3 Hg \left(-\frac{d \ln \rho}{d \ln T} \right) (C_P T)^2 (\nabla_o - \nabla_i)^3 \tag{4.38}$$

By setting $-d \ln \rho / d \ln T = \alpha = 1$, and other values as appropriate for the Sun at radius $r \sim R/2$: $H \sim R$, $C_P T \sim kT\rho/\mu \sim E_T \rho/M$, $\rho \sim M/R^3$, $F \sim L/R^2$, $Hg \sim GM/R \sim (R/\tau_D)^2$ and solving one obtains

$$(\nabla_o - \nabla_i) \sim \left(\frac{\tau_D}{\tau_T} \right)^{2/3} \approx 10^{-7} \tag{4.39}$$

This is a very small value indeed, corresponding to a temperature difference of $1K$ over the total range of temperatures from surface till $T_c \sim 10^7 K$. It justifies use of simplified equation of transport (Eq. 4.32) for convection in deep layers. Substitution of it into Eq. (4.28) yields for deep interior, $\omega_A \sim 2\pi/\tau_D$:

$$\omega_{B-V} \sim 10^{-4} \omega_A \sim \frac{10^{-3}}{\tau_D} \sim 0.1 d^{-1} \tag{4.40}$$

Thus typical time of motion of a bubble τ_{B-V} is of order of a month, long compared to the dynamical time $\tau_D \sim 0.1d$. It justifies neglect of any dynamical effects and validity of hydrostatic equilibrium during the motion.

4.6.3 Radiation losses

Contrary to the case of periodic oscillations, after travelling distance h a bubble does not return, but mixes with its environment and eventually cools down loosing any temperature excess. If so, the cooling time $\tau_C > \tau_{B-V}$. A hot bubble cools down by emitting more flux then receiving from its environment. Heat lost during motion and after cooling down to the environment conditions are proportional to decrease of temperature excess, $\nabla_i - \nabla_{ad}$ and $\nabla_o - \nabla_i$, respectively. Remember that no losses occur only for an adiabatic process $\nabla_i = \nabla_{ad}$ and for no temperature excess in a bubble $\nabla_o = \nabla_i$. Assuming fixed radiative cooling rate over most of life of a bubble we obtain simple proportion

$$\frac{\tau_{B-V}}{\tau_C} = \frac{\nabla_i - \nabla_{ad}}{\nabla_o - \nabla_i} \tag{4.41}$$

An estimate of τ_C is needed prior to use of the proportion.

Excess heat carried by a bubble is $\Delta E \sim H^3 C_P \Delta T$ Heat losses Δl of a bubble may be estimated by adoption of the radiative transport equation Eq. (4.15) and setting $dT/dr \sim \Delta T/H$ and $F \sim \Delta l/H^2$: $\Delta l \sim H^2 F \sim \frac{H\sigma T^3}{\tilde{K}}\Delta T$. Per analogy with Eq. (2.24) we obtain the cooling time:

$$\tau_C \sim \frac{H^2 C_P \tilde{K}}{\sigma T^3} \tag{4.42}$$

Using similar values as for Eq. (4.39) and for the optical depth of a star $\tau \sim \tilde{K}R$ Eddington approximation $\sigma T^4 \sim F\tau \sim LR\tilde{K}/R^2$ one obtains a rough estimate

$$\tau_C \sim \frac{H^2}{R^2}\frac{E_T}{L} \sim \frac{H^2}{R^2}\tau_T \tag{4.43}$$

i.e. many thousands of years.

Using Eq. (4.41) and $\tau_{B-V} \sim h/v \sim \alpha H/v$ we obtain

$$\boxed{\frac{\alpha\sigma T^3}{vHC_P\tilde{K}} \sim \frac{\nabla_i - \nabla_{ad}}{\nabla_o - \nabla_i}} \equiv \frac{1}{\Gamma} \tag{4.44}$$

The final set counts 5 equations (Eqs. 4.34 – 4.37 and 4.44) and 5 unknowns: $\nabla_o, \nabla_i, v, F_r, F_c$. Hence it may be solved for a general case of non-adiabatic convection.

4.6.4 Practical formulae

It is convenient to calculate the solution in dimension-less variables. For completeness we list below required formulae without proof. By setting

$$U \equiv \frac{3acT^3}{\tilde{K}C_P\alpha^2 H^{3/2}}\sqrt{\frac{8}{g\left(-\frac{d\ln\rho}{d\ln T}\right)}}$$

$$W \equiv \nabla_{rad} - \nabla_{ad}$$

$$\xi \equiv \sqrt{\nabla_o - \nabla_{ad} - U^2} \quad \text{one solves}$$

$$(\xi - U)^3 + \frac{8U}{9}(\xi^2 - U^2 - W) = 0 \quad \text{for} \quad \xi$$

where $U \sim \tau_A/\tau_C\alpha^2$ and $\Gamma \sim (\nabla_o - \nabla_i)^{1/2}/2U \sim \tau_C/\tau_{B-V}$

The asymptotic case of $U \to 0 (\Gamma \to \infty)$ corresponds to $\xi \to 0$ and $\nabla_o \to \nabla_{ad}$, i.e. adiabatic convection in deep stellar interior. The asymptotic case of $U \to \infty (\Gamma \to 0)$ corresponds to $\xi \to U$ and $\xi^2 - U^2 = \nabla_o - \nabla_{ad} \to W$, i.e. $\nabla_o \to \nabla_{rad}$, negligible convection.

4.6.5 Role of radiative losses

The to evaluate an effect of radiative losses from a bubble one must consider sign of involved differences. Writing

$$\Delta \ln T = (\nabla_i - \nabla_o)\delta \ln P \tag{4.45}$$

for case of convective stability $\nabla_i - \nabla_o > 0$ and for an initial push inwards, $\delta \ln P > 0$ follows that $\Delta T > 0$. Radiative losses cause loss of heat, hence decrease of ΔT and by consequence of hydrostatic equilibrium (Eq. 4.23) decrease of absolute value of negative $\Delta\rho$. This means that the buoyancy force becomes insufficient to move the bubble back, and resulting amplitude of oscillations is damped. The effect of radiative losses is damping of any oscillations arising in a layer stable against convection. The layer acts as a heat pump: bubbles carry heat upwards at the cost of some mechanical energy.

Interesting possibility arises for a zone of $\Delta_\mu \gg 0$. In such a zone is possible to have $\nabla_i - \nabla_o < 0$ while retaining convective stability $\nabla_i - \nabla_o + (\cdots)\nabla_\mu > 0$. In such a situation a bubble discussed before has $\Delta T < 0$, hence it acquires extra heat resulting in stronger restoring buoyancy force. In effect amplitude of oscillations is rising. The zone acta as a heat engine: it absorbs some transported heat and converts to the mechanical energy.

4.7 Evolution of chemical composition

4.7.1 Initial chemical composition

Composition of young stars is that of the interstellar matter from which they are made. In astrophysics it is customary to express composition by fractions of weight

$$X_i = \frac{\mu_i n_i}{\rho} \qquad \text{where} \qquad \sum_i X_i = 1 \tag{4.46}$$

and n_i is number density of atoms of kind i. Spectroscopic observations of the surface of Sun and related stars revealed composition similar to that of the interstellar matter, in the range indicated in Table 4.3.

4.7.2 Change of composition due to nuclear burning

Let r_{ij} denotes rate per unit volume of nuclear reactions transforming atom i into atom j. If so, the total rate of change of weight content of atom i is

$$\frac{\partial X_i}{\partial t} = \frac{\mu_i}{\rho}\left(\sum_j r_{ji} - \sum_{ik} r_{ik}\right) \tag{4.47}$$

Table 4.3: Surface composition of stars

$X = X_H$	$Y = X_{He}$	$Z = 1 - X - Y$ $\approx X_{CNO}$
$0.6 - 0.7$	$0.36 - 0.3$	$0.04 - 0.001$

The values readily available for astrophysicists are energy production rates

$$\epsilon_{ij} = \frac{1}{\rho} r_{ij} E_{ij} \tag{4.48}$$

where E_{ij} and $Q_i j = E_{ij}/\mu_i$ are energy produced per single reaction and per unit mass. Note that

$$\epsilon = \sum_{ij} \epsilon_{ij} = \frac{1}{\rho} \sum_{ij} r_{ij} E_{ij} \tag{4.49}$$

With these definitions

$$\frac{\partial X_i}{\partial t} = \left(\sum_j \frac{\epsilon_{ji}}{Q_{ji}} - \sum_k \frac{\epsilon_{ik}}{Q_{ik}} \right) \tag{4.50}$$

Particularly

$$\frac{\partial X}{\partial t} = -\frac{\epsilon_H}{Q_H} = -\frac{\partial Y}{\partial t} \tag{4.51}$$

4.7.3 Diffusion

Diffusion is a slow microscopic and selective process of nuclear migration, acting only in absence of macroscopic motions, such as convection. Diffusion is a kind of relaxation process, acting towards restoration of an equilibrium distribution of atoms. One of its effect is dilution of any sharp discontinuities of chemical composition. Another effect is gravity stratification of atoms according to their mass, so that more massive nuclei tend deeper. The corresponding transport equation may be derived similarly to radiation transfer equation. Let v, n and Λ are thermal velocity, number density and mean free path of particles of given kind. Particles from layers in distance Λ mix freely. Their total and net flux accrosc unit surface is

$$\begin{aligned} \text{total flux} \quad & = vn \\ \text{net flux} \quad \mathcal{F} & = v\Delta n \end{aligned}$$

where $\Delta n = \Lambda \partial n/\partial r$. This flux may be expressed as $\mathcal{F} = D\partial n/\partial r$ where $D \sim v\Lambda$ is diffusion coefficient. The corresponding particle conservation equation is $0 = \partial n/\partial t + \partial \mathcal{F}/\partial r$, hence

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial r}\left(D\frac{\partial n}{\partial r}\right) = 0 \tag{4.52}$$

This is diffusion equation for a general case of D variable.

Let h del=notes distance of typical inhomogeneities of particles. Replacing in Eq. (4.52) differentiation with finite differences we obtain expression for diffusion time scale: $1/\tau_{diff} \sim D/h^2$ or

$$\tau_{diff} \sim \frac{h^2}{D} \sim \frac{h^2}{v\Lambda} \sim 10^{13}y \tag{4.53}$$

The last equality holds for the Sun. The corresponding time scale is so long in comparison of the solar age, that diffusion may be safely neglected. However, for a star with $h \sim R/1000$ diffusion time is comparable to the thermal time scale.

The equilibrium distribution of atoms which diffusion is trying to restore is equipartition of particle energies, such that

$$kT \sim E_k \sim E_g \sim g\mu_i H_i \tag{4.54}$$

where H_i is typical scale-height of distribution if particles of kind i. The equilibrium scale heights

$$H_i \sim \frac{kT}{g\mu_i} \tag{4.55}$$

corresponds to more massive atoms tending downwards while less massive atoms tending upwards. Marked effect of diffusion occurs in stars without mixing by convection and with strong gravity acceleration g, namely in white dwarfs.

4.8 Thermal stability

4.8.1 Solar type stars

Stability of thermal or energy equilibrium of a star depends on properties of its nuclear reactor. For present purposes suffices to know that $\partial \ln \epsilon/\partial \ln T >> 0$. Note that at ordinary thermal velocities of nuclei no nuclear fusions takes place while at high temperatures they constitute potent source of energy of thermonuclear explosions. Hence nuclear reaction rate grows with temperature, $\partial \ln \epsilon/\partial \ln T > 0$.

Perturbations of energy equilibrium evolve on thermal time scale, τ_T, slow compared to the dynamic time scale τ_D. For this reason hydrostatic equilibrium is maintained during the evolution. For the Sun and similar stars built of near perfect gas the following form the virial theorem holds (Eq. 2.22): $-E \sim \frac{GM^2}{R} \sim \frac{M}{\mu}kT$. In the differential form:

$$\delta \ln(-E) = -\delta \ln R = \delta \ln T \tag{4.56}$$

From the continuity equation $\rho \sim M/R^3$ i.e.

$$\delta \ln \rho = -3\delta \ln R \tag{4.57}$$

For a rough estimate we replace in the energy conservation equation (Eq. 4.1) derivatives dl and dr with finite differences, set $\rho \sim M/R^3$ and obtain in this way

$$L \sim M\epsilon. \tag{4.58}$$

Such an approximation is valid for the Sun and similar stars in which parameters vary smoothly with radius.

Equation (4.58) maintains balance of losses and sources of energy in a star. Perturbation of the balance results in gradual change of the total energy E of a star by δE:

$$\frac{d\delta E}{dt} \sim \delta(M\epsilon - L) \sim L(\delta \ln \epsilon - \delta \ln L) \tag{4.59}$$

Note use of Eq. (4.58) in the last transformation. Now let us exploit Eq. (4.56):

$$\frac{d\delta \ln T}{dt} = \frac{d\delta \ln(-E)}{dt} \sim \frac{1}{-E}\frac{-d\delta E}{dt} = \frac{L}{-E}(\delta \ln L - \delta \ln \epsilon)$$

It is worth recalling here the thermal time scale $\tau_T = -E/L$ (Eq. 2.24). The only strong dependence on the R.H.S. is that of ϵ on T. By exploiting it we obtain the following approximation:

$$\frac{d\delta E}{dt} \sim -\frac{1}{\tau_T}\frac{\partial \ln \epsilon}{\partial \ln T}\delta \ln T \tag{4.60}$$

Eq. (4.60) constitutes a first order linear differential equation for evolution of the perturbation $\delta \ln T$. Substitution of a trial solution $\delta \ln T \sim e^{\omega t}$ for $\omega < 0$ yields the stability criterion of thermal balance:

$$\omega =\sim -\frac{1}{\tau_T}\frac{\partial \ln \epsilon}{\partial \ln T} < 0 \tag{4.61}$$

Since $\frac{\partial \ln \epsilon}{\partial \ln T} > 0$, the criterion (Eq. 4.61) is fulfilled, we conclude with relief that our Sun is stable against perturbation of nuclear reactions. In consistency with our discussion of Eq. (2.24) the time scale of evolution of perturbations of energy balance is τ_D.

Let us investigate the effect of the neglected dependence of L and ϵ on their parameters. Writing the equation of radiation transport as $L/R^2 \sim F \sim (d\sigma T^4/dr)/\tilde{\kappa}\rho \sim \sigma T^4 R^2/\tilde{\kappa}M$ and differentiating we obtain

$$\delta \ln L = 4\delta \ln R + 4\delta \ln T - \delta \ln \tilde{\kappa} = -\delta \ln \tilde{\kappa} \tag{4.62}$$

Deep in stellar interior, where gas is completely ionized $\delta \ln \tilde{\kappa}$ may be neglected.

Dependence of ϵ on T was accounted properly. Hence it remains to discuss similar dependence on ρ. The rate of fusions per volume depends on rate of colisions, i.e. on number of pairs of particles $\sim \rho^2$. Hence the rate of reactions per mass $\epsilon \sim \rho^1$, i.e. $\partial \ln \epsilon/\partial \ln \rho = 1$. For the complete difference one obtains

$$\delta \ln \epsilon = \frac{\partial \ln \epsilon}{\partial \ln T}\delta \ln T + \frac{\partial \ln \epsilon}{\partial \ln \rho}\frac{d \ln \rho}{d \ln R}\frac{d \ln R}{d \ln T}\delta \ln T = \left[\frac{\partial \ln \epsilon}{\partial \ln T} + (+1)(-3)(-1)\right]\delta \ln T \tag{4.63}$$

From combination of both results follows that

$$\delta \ln L - \delta \ln \epsilon = 4\delta \ln R + 4\delta \ln T = -\left(\frac{\partial \ln \epsilon}{\partial \ln T} + 3\right)\delta \ln T - \delta \ln \tilde{\kappa} \tag{4.64}$$

Such a result indicates that the neglected effects tend to make the Sun even more stable.

4.8.2 Barotropic stars

Stability of stars with barotropic equation of state constitutes a case different from the solar type stars. The barotropic equation of state has the form $P = P(\rho)$. Now the total energy of a star does depend on P but it does not depend on T.

$$-E \sim \frac{GM^2}{R} \sim PR^3 \quad \text{where} \quad \frac{\partial P}{\partial T} \equiv 0 \tag{4.65}$$

$$E_T \sim \frac{M}{\mu}kT << PR^3 \tag{4.66}$$

Nuclear burning produces heat and increases E_T and T. However, since P does not depend on T, and $E_T \ll |E|$ the total energy remains almost unaffected. So does radius R and ρ. Thus situation resembles heating of a piece of metal: The temperature grows, but no change occurs in its structure. The new perturbation equation is obtained from the previous one by substitution of $-E$ for E_T

$$\frac{d\delta \ln T}{dt} = \frac{d\delta \ln E_T}{dt} \sim -\frac{L}{E_T}(\delta \ln L - \delta \ln \epsilon) \sim +\frac{1}{\tau_T}\frac{\partial \ln \epsilon}{\partial \ln T}\delta \ln T \tag{4.67}$$

The stability criterion in this case is

$$\omega = +\frac{1}{\tau_T}\frac{\partial \ln \epsilon}{\partial \ln T} < 0 \tag{4.68}$$

For $\frac{\partial \ln \epsilon}{\partial \ln T}$ the criterion is violated.

The barotropic stars are thermally unstable. The result of starting of reactions is exponential growth of their rate and a thermonuclear runaway explosion. The effect resembles starting reactions in a solid container, i.e. a thermonuclear bomb. The explosion may stop no sooner than for $E_T \sim E$, when ordinary perfect gas pressure becomes important. Barotropic equations of state operate in degenerate cores of red giant stars and in white dwarfs. In red giants ignition of helium in a degenerate core, so called helium flash, generates thermal instability manifested as loops of red giant evolutionary tracks. In cataclysmic binaries helium or or carbon white dwarf stars accrete from their companions and in this way build a hydrogen rich shell. Ignition of hydrogen burning in a degenerate layer of the shell results in a thermonuclear explosion of a shell, manifested as a nova explosion. On the contrary supernova outburst are caused by dynamical instability of the core and thermonuclear explosion is only a consequence and not cause of the instability.

4.9 Stellar structure equations

4.9.1 The full set of equations

Mathematically equation of stellar structure are simple system of 4 first order ordinary differential equations. However, their right hand sides involve state functions, which are extremely complex functions of parameters of state, which only available representation are extensive tables stored on computer media. For this reason realistic solutions may be obtained only by numerical methods. For these methods it is important that integrated solutions are smooth functions of independent variable. In this respect mass variable m and not radius r is more suitable independent variable of integration. Thus for practical use Eq. (2.1) is transformed to $dr/dm = (dm/dr)^{-1}$ and other equations are divided by Eq. (2.1) to yield derivatives with respect to m. In this way Eqs. (2.1), (2.4), (4.1), (4.32) and (4.50) assume the following 'professional' form:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{4.69}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \boxed{\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}} \qquad \tau_D \tag{4.70}$$

$$\frac{\partial l}{\partial m} = \epsilon_n - \epsilon_\nu - \boxed{\frac{\partial Q}{\partial t}} \qquad \tau_T \tag{4.71}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla \tag{4.72}$$

Table 4.4: State functions of stellar structure

Functions	Process
$\rho, \dfrac{\partial \ln \rho}{\partial \ln T}, \dfrac{\partial \ln \rho}{\partial \ln \mu}$	equation of state
∇_{ad}, c_P	e.s. & thermodynamics
$\tilde{\kappa}$	opacity
$\epsilon_n, \epsilon_\nu, r_{jk}$	nuclear reactions

$$\boxed{\frac{\partial X_i}{\partial t} = \left(\sum_j \frac{\epsilon_{ji}}{Q_{ji}} - \sum_k \frac{\epsilon_{ik}}{Q_{ik}} \right)} \quad \tau_N \tag{4.73}$$

In the transport equation symbol ∇ indicates either ∇_{rad} Eq. (4.35) or ∇_{ad} Eq. (4.32), depending whether energy is transported by radiation or convection. For the radiation transport in Eq. (4.35) one sets $F_r \equiv l/4\pi r^2$.

In the equations the terms reflecting effects of time are framed and associated time scales are indicated in the right margin. Since $\tau_D << \tau_T << \tau_N$, usually no more then one of the terms plays significant role. Other processes either have enough time to reach equilibrium or insufficient time to produce any significant effects. The time effects, i.e. terms involving $\partial/\partial t$ are usually small. In calculations they are neglected or assumed known from previous models. In this way the stellar structure and evolution problem may be treated as consecutive solution of equations of state (Eqs. 4.69 - 4.72), which may be considered single first order differential equation in the space of vectors.

$$\overline{y} = (r, P, l, T). \tag{4.74}$$

Stellar evolution is calculated in the following way: first a model is calculated for an assumed initial chemical composition, by solution of Eqs. (4.69 - 4.72). Then a sufficiently small time step Δt is selected to calculate change of chemical composition $X_i(m, t + \Delta t) = X_i(m, t) + (\partial X_i(m, t)/\partial t)\Delta t$ (Eq. 4.73) in every layer of a star. Next evolutionary model is calculated for the new composition, then change of composition is calculated again, and again.

4.9.2 State functions

Complex stellar physics is contained in in the state functions describing local conditions of the stellar matter. They are functions of parameters of state such as pressure, temperature and chemical composition $P, T, X, \cdots$. A complete list of state functions is displayed in Table 4.4. These functions of state suffice for finding solutions of stellar structure equations. In addition to them for purpose of computational convenience and acceleration of calculations are required derivatives of the R.H.S. of equations, involving derivatives of the state functions, over dependent variables $\overline{y} = (r, P, l, T)$ (Eq. 4.74). Quality of these derivatives affects speed of convergence of the solutions but not physics of the converged solutions. It often suffices to calculate the latter derivatives by approximate numerical differentiation.

For solar type main sequence stars equation of state accounts for perfect gas equation including calculation of ionization of basic gas components. It involves effects of partial degeneracy of electrons. For calculation of stellar evolution equation of state ought to treat partial and full degeneracy in classical and relativistic regimes. Functions ∇_{ad}, c_P, i.e. the adiabatic gradient and specific heat are computed from the equation of state by means of thermodynamic relations. In main sequence stars energy production ϵ_n accounts for hydrogen burning in $p-p$ and CNO cycles. For advanced evolution ϵ_n accounts for further reactions, e.g. 3α, CO burning. Energy losses due to emission of neutrinos are accounted in ϵ_ν. Complex as these functions of state are, they are simple enough to be calculated on-the-fly during calculations. Investigation of special effects, such as synthesis of many isotopes X_i and cosmochemistry requires detailed accounting for multiple reaction chains and their rates r_{ij}, possibly in collaboration with concerned nuclear physicists.

By far the most complex state function is opacity κ. Since long time opacity calculations involved continuum effects from bound-free and free-free states of dozens of ions and their base and excited states. Yet recent calculations by two large groups (OPAL in Lawrence Livermore Labs and an international collaboration OP) revealed that contribution by myriads of bound-bound transitions (absorption lines) in exotic ions, e.g. of the iron group raise κ by a factor of several. This affects importantly stellar properties. The state of art opacity calculations became an industry of their own as they involve quantum mechanical calculations of properties of millions of atomic levels and configurations. In modern stellar calculations opacity is calculated by interpolation of extensive tabulations of results distributed by OPAL and OP consortia over internet in the computer readable form.

4.9.3 Boundary conditions: the professional version

In general form the boundary conditions may be written in terms of 4 functions b_j of parameters $\overline{y}$ (Eq. 4.74):

$$0 = b_j(\overline{y}_0) \quad \text{for} \quad j = 1, 2 \quad \text{at center} \tag{4.75}$$
$$0 = b_j(\overline{y}_M) \quad \text{for} \quad j = 3, 4 \quad \text{at surface} \tag{4.76}$$

Solution of a system of differential equations is unique a Cauchy problem, i.e. for the boundary conditions all set at the same place. The boundary conditions for stellar structure are not fixed at a single location, hence the corresponding problem is not of Cauchy type. The corresponding problem is an eigenvalue problem. Any solutions are guaranteed to be separated by finite amounts. Non unique solutions are known, as a rare curiosity of no practical implications.

Boundary conditions at center

The original boundary conditions at center $m = l = 0$ for $r = 0$ are inconvenient for calculations at R.H.S. of Eqs. (4.69) - (4.72) suffer from $0/0$ singularities at $r = 0$. For this reason it is necessary to use analytical expansions to propagate solutions from $m_0 = 0$ to $m_1 \ll M$. The central region is approximated by a sphere of constant density ρ_c and mass m_1. The corresponding pair of the boundary conditions near center is

$$r_1^3 = \frac{3}{4\pi\rho_c} m_1 \tag{4.77}$$

$$l_1 = \left(\epsilon_n - \epsilon_\nu + \frac{\partial Q}{\partial t}\right)_c m_1 \tag{4.78}$$

Solutions yield directly P and T at m_1. Additional formulae are required to calculate $f_c = (P_c, T_c)^T$. They use Taylor expansion at 0, accounting for the extremum conditions $df/dm_c = 0$: $f(m_1) = f_c + 0 + (m_1^2/2)(d^2f/dm^2)_c$. Result of differentiation over m_1 is $df/dm_1 = m_1(d^2f/dm^2)_c$. After elimination of $(d^2f/dm^2)_c$ the expansion yields

$$\begin{pmatrix} P \\ T \end{pmatrix}_1 = \begin{pmatrix} P \\ T \end{pmatrix}_c + \frac{m_1}{2}\frac{d}{dm}\begin{pmatrix} P \\ T \end{pmatrix}_1 \tag{4.79}$$

Boundary conditions at surface

The provisional boundary conditions at surface $P \approx T \approx 0$ for $m = M$ are to rough to be used in actual models. A better boundary condition is obtained by assuming $\kappa \equiv const$ and integration of the hydrostatic equilibrium equation times κ:

$$\boxed{P\kappa =} \kappa \int_r^\infty \frac{dP}{dr} dr = g \int_r^\infty \kappa\rho dr = \boxed{g\tau} \tag{4.80}$$

Another boundary condition is obtained by integration of Eq. (4.13) $(4\pi/3\tilde{K})(dB/dr) = F \equiv const$ over optical depth $d\tau = \tilde{K}dr$:

$$\boxed{T^4 = \frac{3}{4}(T_e^4\tau + q)} \tag{4.81}$$

where $\pi B = \sigma T^4$, $L/4\pi R^2 = F \equiv \sigma T_e^4$ and $q \approx 2/3$. This is called Eddington approximation for the stellar atmosphere. In Eqs. (4.80) and (4.81) $\tau \gg 1$ is a free parameter indicating optical depth where the boundary condition is applied.

The boundary conditions (Eqs. 4.80 and 4.81) are good enough for many purposes. However, sometimes more accurate conditions are required. These are obtained from the computed stellar atmosphere models. The atmosphere models depend on two parameters T_e and $g = GM/R^2$. These two parameters and continuity conditions at the optical depth τ_M yield the boundary conditions in the general form of Eq. (4.76):

$$0 = b_j(\overline{y}_M)_{\tau_M} \qquad \text{for} \quad j = 3, 4$$

4.10 Computation methods

As an introduction we start by recalling Newton method of numerical solution of equations. Later we shall proceed to discussion of two basic methods used for calculation of stellar models, namely Schwarzschild and Henyey methods.

4.10.1 Newton-Raphson iterations

Newton method for solution of equations is usually taught at the basic course of numerical methods. Let y_0 indicates an initial guess for the solution and the equation has the form

$$0 = f(y) \tag{4.82}$$

Introducing δy such that $y = y_0 + \delta y$, the equation becomes $0 = f(y_0 + \delta y)$ Taylor expansion of its R.H.S. around y_0 truncated after second term yields

$$0 = f(y_0) + \frac{df}{dy}_0 \delta y \tag{4.83}$$

This is a linear equation for δy. By solving it we find new solution $y_0 + \delta y$. This solution is not accurate, for higher terms of Taylor expansion were neglected. However, for a sufficiently close initial guess, higher terms could be quite small and the new solution $y_0 + \delta$ would be better than y_0. Now substitution of $y_0 + \delta$ for a new initial guess and so on results in iterations which may converge to a solution. In the rest of this section y and $f(y, m)$ indicate dependent variables of integration and R.H.S. of differential equations. Note use of symbol m for dual purposes, which should be clear from its context: as a mass variable and for indexing stellar layers. Thus m_{m+1} is mass variable corresponding to the layer number $m + 1$.

4.10.2 Schwarzschild method

Possibly simplest idea on how to solve the structure equations is to supplement two known parameters at one end with 2 more guessed (say P_c and T_c at $m = 0$) and to integrate the equations from this initial value up to the other end to check whether the other 2 boundary conditions are satisfied. Unfortunately this trial-and-error or one-side shooting method fails badly. The integrated solutions are unstable in the sense that they diverge so wildly before reaching the other end, that due to rounding errors it is impossible to calculate a solution remotely approaching conditions required at the other end. Did you ever try to put through a needle one end of a thread while keeping the wrong end of a thread?

The shooting method was worth of mentioning, however, for its two side cousin proved to work well. In this method 2×2 boundary parameters are guessed at both ends, $m = 0, M$ and integrations are carried from both sides towards an intermediate layer of a star:

$$\begin{aligned} y_-(M/2) &= y_0 + \int_0^{M/2} f dm \\ y_+(M/2) &= y_M + \int_M^{M/2} f dm \end{aligned}$$

At this layer, say $M/2$ we impose smooth joining condition for $y_\pm$ at 4 variables, enough to fix right initial guesses:

$$0 = y_-(M/2) - y_+(M/2) \tag{4.84}$$

Eq. (4.84) expanded in a truncated Taylor series yields

$$0 = y_- - y_+ + C_-\delta y_0 - C_+\delta y_m \tag{4.85}$$

where the derivatives $C_\pm = \partial y_\pm / \partial y_{0,M}$ may be approximated from finite differences based on trial calculations. Only 4 of components of δy_0 and δy_m are not vanishing, hence after first shot, solution of Eq. (4.85) yields a set of corrections δy. The corrected solution is not exact, for we neglected higher orders of Taylor expansion. The corrected solution is used as intelligent guess for the next iteration. For sufficiently accurate first guess iterations converge to a solution. This elaborated two-side-shooting integration method in astronomy is called Schwarzschild method.

4.10.3 Henyey relaxation method

Schwarzschild method was formulated in analytical terms and numerical methods were employed only later in integrations. In another method due to Henyey the numerical aspect is introduced first and analytical treatment follows. A star is sliced in several hundred of layers numbered with $m = 1, \cdots M$. The differential equations are replaced with a set of $M - 1$ finite difference equations between adjacent layers m and $m + 1$. This is the numerical aspect of the method. So the correct solution y should satisfy the system of equations

$$0 = \frac{y_{m+1} - y_m}{m_{m+1} - m_m} - \frac{f(y_{m+1}) + f(y_m)}{2} \tag{4.86}$$

for $m = 1, \cdots, M - 1$, corresponding to $4 \times (M - 1)$ equations. In addition to equations (e15.3) there are 4 boundary conditions (Eqs. 4.75 and 4.76). In total there are $4M$ equations for the same number of unknowns $\delta y_m, m = 1, \cdots M$.

To simplify notation these equations are written as

$$\begin{array}{ll} 0 = c_0(y_1) & 2 \quad \text{equations} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 = c_m(y_m, y_{m+1}) & 4 \times (M - 1) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 = c_M(y_M) & 2 \end{array} \tag{4.87}$$

where c-s are known functions of y-s.

An initial guess of the values of the solution function $y(m)$ for each layer is required to start calculations. In general a guessed solution does not satisfy Henyey equations (Eq. 4.87). If so, the R.H.S of Eqs. (4.87) do not vanish for y. Hence, similarly as for Schwarzschild method, equations are expanded around y into a truncated Taylor series. Let $y + \delta y$ denotes the searched for solution of the Henyey equations of structure. The expanded equations yield linear algebraic equations for values of the correction function δy:

$$\begin{array}{llll} 0 = c_0(y_1) & +C_0 \delta y_1 & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ 0 = c_m(y_m, y_{m+1}) & & +C_{-m}\delta y_m \quad +C_{+m}\delta y_{m+1} & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ 0 = c_M(y_M) & & & +C_M \delta y_M \end{array} \tag{4.88}$$

where c_m and $C_{\pm m} = \partial c_m / \partial y_{m+1/2 \pm 1/2}$ are vectors of R.H.S. of Eqs. (4.87) and matrices of their derivatives computed for the guessed solution y. This is a linear system for $4M$ unknown corrections δy_m, i.e. 4 values of y for each layer. Formally this system, $0 = l + H \delta y$ is huge, for H is a $4M \times 4M$ Henyey matrix. However, it is not difficult to see that all but coefficients nearest of the diagonally vanish.

4.10.4 Three-diagonal elimination

Solution of a general system with matrix H of size $N = 4M$ requires $\mathcal{O}(N^3)$ operations, a huge number for typical case of $4M > 1000$. For this reason it is important to exploit the three-diagonal block symmetry of matrix H. Matrix H partitioned into 4×4 matrices of coefficients, called blocks, contains non-zero blocks only on its diagonal and next sub- and super diagonal. For systems with such matrixes exists efficient algorithm requiring $\mathcal{O}(\alpha N)$ calculations for solution. Here α is number of operations for inversion of a block, i.e. $4^3 = 64$ in the present case. It is worth noting that rules for combination of $+, -, *, / \equiv (\cdot)^{-1}$ matrix operations on blocks are the same the same rules as for ordinary operations on numbers. If so, an algorithm derived for an ordinary three-diagonal matrix may be trivially adapted for a block matrix by simple substitution of corresponding matrix operations.

Solution of a three-diagonal system is performed in two steps. The first step is to transform the system by elimination to a two diagonal one:

$$
\begin{array}{c|cccccc}
l_1 & A_1 & B_1 & & & & \\
l_2 & A_2 & D_2 & B_2 & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l_m & & A_m & D_m & B_m & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l_{M-1} & & & A_{M-1} & D_{M-1} & B_{M-1} & \\
l_M & & & & A_M & D_M &
\end{array}
\Rightarrow
\begin{array}{c|ccccc}
l'_1 & I & B'_1 & & & \\
l'_2 & & I & B'_2 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l'_m & & & I & B'_m & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l'_{M-1} & & & & I & B'_{M-1} \\
l'_M & & & & & I
\end{array}
\tag{4.89}
$$

where I denotes either number 1 or unit matrix. The next step is solution of the two-diagonal system. Both steps may be compactly written as two program loops:

$$
\begin{array}{lll}
A_1 & = B_M = 0 & \text{initialization} \\
\hline
D'_m & = D_m - A_m B'_{m-1} & \\
B'_m & = D'^{-1}_m B_m & \text{elimination} \\
l'_m & = D'^{-1}_m (l_m - A_m l'_{m-1}) & \quad \text{for} \quad m = 1, \cdots, M \\
\hline
\delta y_m & = -B'_m \delta y_{m+1} + l'_m & \text{backsubstitution} \\
y_m & = y_m + \delta y_m & \quad \text{for} \quad m = M, \cdots, 1
\end{array}
\tag{4.90}
$$

Formulae (4.90) may be proven by induction. Let all rows of H up to $m-1$ are already in the 2-diagonal form and remaining in the 3 diagonal form. Hence $m-1$-th row reads $l'_{m-1} = y_{m-1} + B'_{m-1} y_m$. Solution of it for y_{m-1} and substitution into m-th row $l_m = A_m y_{m-1} + D_m y_m + B_m y_{m+1}$ yields $l_m = A_m(l'_{m-1} - B'_{m-1} y_m) + D_m y_m + B_m y_{m+1}$ and after reordering of terms $l_m - A_m l'_{m-1} = (D_m - A_m B'_{m-1}) y_m + B_m y_{m+1}$. Comparison of respective terms reveals that D'^{-1}_m times the last equation is equivalent to the 2-diagonal form $l'_m = y_m + B'_m y_{m+1}$ if D'_m, l'_m and B'_m are defined as in Eq. (4.90).

For the Henyey systems of equations, l and l' are vectors of length 4 while A, B, D, I, B' and D' are matrices 4×4. Note that the primed values l', B' and D' may be stored in place of l, B, D. Typically for Newton method, the new solution $y + \delta y$ is not accurate enough because of neglect of higher order terms of Taylor expansion. It may be considered a new guess for the next iteration of the Henyey solution. Iterations must be continued till l and δy become sufficiently small. As usual for Newton method, convergence of iterations is guaranteed only for an initial guess close enough to the solution. In this respect Henyey method in application to stellar structure is quite demanding. For this reason inefficient Schwarzschild method is often used to compute approximate solution to start Henyey calculations. Note that the decision whether y is a solution of Henyey equations depends on vanishing of $c(y)$. For this reason accuracy of derivatives C in Eq. (4.89) does not affect accuracy of the solution. Obviously it does affect convergence to the solution since for wrong C one obtains wrong δy. Still, it is often enough to approximate C with finite differences of c, i.e. with numerical derivatives $\Delta c/\delta y$. This is a great simplification, for c involve tabulated functions of state with no derivatives known.

Power of Henyey method is best seen in calculation of sequences of stellar models. For example stellar evolution is calculated as a sequence of models with chemical composition of the next one slightly modified by nuclear reactions as compared to the previous one. For the Henyey method a previous stellar model constitutes good initial guess for next one. For good choice of Δt one or two Henyey iterations suffice for each model and hundred of models is enough to calculate evolution from the main sequence to the red giant phase. In a similar way Henyey method may be efficiently used for calculation of a sequence of models with the same composition X but slightly different mass M, and so on.

4.10.5 Stability of modelled stars

It may be instructive at this point to deviate attention from computations in order to discuss stability of stars corresponding to calculated stellar models. By definition of neutral equilibrium, a star pushed slightly away from its neutral equilibrium state remains in another equilibrium state. In other words neutral stability requires existence of a family of infinitesimally close equilibrium states for a star of the same mass. In formulation of Henyey and Schwarzshild methods this requires a family of different solutions δy satisfying the same system H or C, respectively. For inhomogeneous systems of linear algebraic equations such a case may happen if and only if $detH = detC = 0$. Thus the determinants must vanish simultaneously and only for stellar models in neutral equilibrium. With little more insight it is possible to see that in a continuous family of stars corresponding, say, to different total mass, stars in neutral equilibrium will separate stars in stable and unstable equilibrium. So sign of $detH$ or $detC$ indicates type of stability the calculated stellar model. Note that no real star may exist in an unstable equilibrium. This has practical consequences in calculations. Namely, H does not become singular for realistic models hence all block matrices to be inverted are not singular.

Chapter 5

State functions

5.1 Equations of state

Newton second law of dynamics may be written as $d(mv)/dt = F$. Hence pressure may be expressed as

$$P \sim \frac{\textit{force}}{\textit{surface}} \sim \frac{\textit{flux of momentum}}{\textit{time} \cdot \textit{surface}} \tag{5.1}$$

This kind of relation constitutes basis of derivation of equations of state.

5.1.1 Radiation

In stellar interior photon mean free path is as short as $\Lambda = 1/\kappa \sim 1cm$ (for the Sun), so radiation has little chance to escape and its properties approach the black body radiation for thermodynamic equilibrium. Pressure and energy density of the black body radiation is (Eqs. 2.12 and 2.13). Other functions of state follow from simple differentiation:

$$\boxed{U = \frac{4\sigma T^4}{c}} \qquad \boxed{P_r = \frac{4\sigma T^4}{3c}} = \frac{1}{3}U$$

$$\frac{d\ln T}{d\ln P} = \boxed{\nabla_{ad} = \frac{1}{4}} \qquad \boxed{\gamma \equiv \frac{1}{1-\nabla_{ad}} = \frac{4}{3}}$$

$$C_V = \frac{d\ln U}{d\ln T} = \frac{16\sigma T^3}{c} \qquad C_P = \gamma C_V = \frac{64\sigma T^3}{3c}$$

Note that in this case capitalized letters indicate values per unit volume.

5.1.2 Perfect gas

For perfect gas kinetic energy of particles defines the temperature scale $\mu v^2 \sim kT$. Here v^2 is mean squared velocity of thermal motions. Let us consider a cube of side l with number density of enclosed particles n. Only $n/6$ of particles is going to hit a specific wall of the cube. Among those, only particles within distance $< 1 \cdot v$ of the wall bounce from it within unit time. These particles come from the volume vl^2 hence flux of particles per unit surface is $vn/6$. For elastic bouncing particles change momentum from $+\mu v$ to $-\mu v$, i.e. by $2\mu v$, on average. Substitution into Eq. (5.1) after neglection of factors of order of unity yields pressure

$$\boxed{P \sim} \mu v v n \sim nkT \sim \boxed{\frac{\rho}{\mu kT}} \sim \frac{m}{V}\frac{\mu_H}{\mu}\frac{k}{\mu_H}T \sim \mathcal{M}\mathcal{R}\frac{T}{V} \tag{5.2}$$

where $\mathcal{M} = m/(\mu/\mu_H)$ is number of moles of gas and $\mathcal{R} = k/\mu_H$ is gas constant.

In stellar interiors atoms of several kinds contribute to pressure, so

$$n = \sum_i n_i = \rho \sum_i \frac{X_i}{\mu_i} \tag{5.3}$$

where X_i is fractional content of i-th atom, by weight. Average particle mass is ρ/n, hence

$$\boxed{\frac{1}{\mu} = \sum_i \frac{X_i}{\mu_i}} \equiv \frac{1+E}{E\mu_e} \equiv 1 + E\mu_a \tag{5.4}$$

where $E = n_e/n_a$ is fraction of numbers of free electrons-to-atoms and μ_e and μ_a are average mass of gas per electron and atom, respectively. For hydrogen, helium and CNO content X, Y and Z, after full ionization contributing 2, $3/4$ and $1/2$ particles per proton mass, the average masses are:

$$\begin{aligned}
\frac{1}{\mu} &= (2X + \frac{3}{4} + \frac{1}{2}Z)\frac{1}{\mu_H} \\
\frac{1}{\mu} &= (X + \frac{1}{2} + \frac{1}{2}Z)\frac{1}{\mu_H} \\
\frac{1}{\mu} &= (X + \frac{1}{4} + 0)\frac{1}{\mu_H}
\end{aligned}$$

Energy density and specific heats per unit mass are

$$\boxed{u = \frac{3}{2}\frac{kT}{\mu}} \qquad c_V = \frac{u}{T} = \frac{3}{2}\frac{k}{\mu} \qquad c_P = \frac{u+pV}{T} = \frac{5}{2}\frac{k}{\mu}$$

$$\gamma \equiv \frac{c_P}{c_V} = \frac{5}{3} \qquad \boxed{\nabla_{ad} = \frac{\gamma-1}{\gamma} = \frac{2}{5}}$$

5.1.3 Ionization

Boltzmann and Saha distributions

According to Boltzmann law particles of a given type in thermodynamic equilibrium are so distributed among energy states, that the ratio of their population n_1 and n_2 over states of energies E_1 and E_2 is

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-\frac{E_1 - E_2}{kT}} \tag{5.5}$$

Here g_1 and g_2 are statistical weights of the corresponding states 1 and 2. The weights indicate the total number of realizations of a given state. A similar formula holds for the ratio of population one state n_1 to the total number of given particles:

$$\boxed{\frac{n_1}{n} = \frac{g_1}{u} e^{-\frac{E_1}{kT}}} \tag{5.6}$$

The similarity is only formal, as u called a partition function is a sum of statistical weights of all energy levels, say all states of an atom, weighted by probability of their population. There are infinite many energy levels in an atom, so a straight sum of their statistical weights is always infinite. However, very high states easily extend to dimensions excideeng distance to other atoms. These states may never be populated and the weighted summation is truncated in a way depending on gas state, so that $u = u(\rho, T)$ is a state function.

Ionization of atom A, $A_i \leftrightarrow A^+_{i+1} + e^-$, may be considered as transition of an electron from a bound state of energy $-\chi_i$ up to one of the continuum of free states, with energy $p_e^2/2m_e$, where χ_i is ionization energy, p_e and m_e are electron momentum and mass. Applying Boltzmann law we obtain

$$\frac{dn_{i+1}}{n_i} = \frac{u_{i+1}}{u_i} dg_e e^{-\frac{p_e^2/2m_e + \chi_i}{kT}} \tag{5.7}$$

Note that the statistical weight of the upper state is product of the weights of ion g_{i+1} and its electron dg_e for a range of momentum dp_e. Electrons are fermions, i.e. one state may be occupied by no more than one electron. According to the uncertainity principle, a state of an electron occupies at least volume $\Delta x^3 \Delta p^3 > h^3$ in the phase space of momentum and space volumes. If so the number of available states $dg_e \sim d^3x d^3p/h^3 \sim p_e^2 dp_e/n_e h^3$ where $1/n_e$ is volume per electron and $p_e^2 dp_e$ is d^3p for spherical symmetry. Substitution into Eq. (5.7) and change of variable $y^2 = p_e^2/m_e kT$ yields

$$\frac{dn_{i+1}}{n_i} \sim \frac{u_{i+1}}{u_i} e^{-\frac{\chi_i}{kT}} \frac{(m_e kT)^{3/2}}{n_e h^3} e^{-y^2} y^2 dy \tag{5.8}$$

Integration of this equation results in Saha equation

$$\boxed{\frac{n_{i+1}}{n_i} = 2\frac{u_{i+1}}{u_i}\frac{(2\pi m_e kT)^{3/2}}{n_e h^3} e^{-\frac{\chi_i}{kT}}} \tag{5.9}$$

where the weight factor of 2 accounts for two possible spin orientations.

Ionization balance

Writing R.H.S. of the Saha equation for atom of type j as $f_{ji}(n_e, T)$ one obtains the following system of equations for population of the ionization states:

$equation$	$number$	$unknowns$
$n_{ji+1} = n_{ji} f_{ji}(n_e, T)$	$J(I-1)$	n_{ji}
$\sum_i n_{ji} = \rho \frac{X_j}{\mu_j}$	J	n_{j1}
$\sum_{ij} i n_{ji} = n_e$	1	n_e

(5.10)

In practice, it is best to select for unit of n_{ji} population of the most populated state and only later to find value of the unit from $\sum_i n_{ji}$. This state is easy to find by change of sign in the sequence of $\ln f_{ji}$ for $j = 1, \cdots, J-1$. The perfect gas equation of state taking into account ionization is

$$\boxed{P = (n_e + \sigma_{ji} n_j i)kT} \tag{5.11}$$

Note that near thermodynamic equilibrium average energy of particles of all kinds is kT. This is called equipartition of energy.

Adiabatic gradient

In a range of temperatures where ionization is partial, number of particles of gas varies with degree of ionization and dependence $P(T, V)$ is complex. However, for low and high temperatures ionization is either negligible or complete and the corresponding plots of $\ln P$ against $\ln T$ for an adiabatic process asymptotically are parallel straight lines of inclination $1/\nabla_{ad} = 5/2$. The line for high temperature, corresponding to full ionization lies higher as more particles are involved (Eq. 5.11). For rising temperature, in the intermediate regime the P-T relation jumps up to the higher line, i.e. its inclination is steeper than $5/2$. Swopping axes of the plot reveals that in the partial ionization regime the adiabatic gradient must be less than the canonical value of $2/5$, namely $0 < \nabla_{ad} < 2/5$.

At conditions of phase transition in a substance, change of pressure leaves temperature constant but affects balance between two phases. In a sense partial ionization does resemble phase transition except that change of pressure results in slow change of temperature but it never remains constant.

Pressure ionization

From average volume per atom one could derive average atom separation as $d \sim (\mu/\rho)^{(1/3)}$. Electron in distance d from its atom becomes no longer bound. Effectively the ionization potential becomes lowered by $\Delta\chi \sim Ze^2/d$. This may produce *pressure ionization* as soon as $\Delta\chi$ becomes comparable with the ionization potential χ itself. Solving these two equalities we obtain the critical density for setting in of the pressure ionization

$$\rho \sim \mu \left(\frac{\chi}{Ze^2} \right) \sim 10^3 g/cm^3 \tag{5.12}$$

Density in solar type stars is just a bit less. In practice by combined effects of high P and T s olar interior gets completely ionized.

5.1.4 Degeneracy

Pressure of degeneracy

Electrons as particles with fractional spin $1/2$ are fermions. As such they obey Pauli exclusion principle: no more than one electron may occupy a given quantum state. Heisenberg indeterminacy principle yields the minimum volume of the phase space occupied by a state:

$$\Delta x^3 \Delta {p_x}^3 \geq h^3. \tag{5.13}$$

The average volume per electron is μ_e/ρ where μ_e is mean mass of gas particles per each electron. The macroscopic consequence of Pauli and Heisenberg principles is inability of electron gas to reduce at the same time both its volume and momentum per particle below certain value $(\mu_e/\rho)p_F^3 = h^3$. The minimum configuration corresponds to all low momentum states $|p_x| < p_F$ being filled and no electron left with extra momentum. The maximum value of momentum and corresponding electron energy are called Fermi momentum end energy p_F and E_F. Clearly, both p_F and E_F depend on density of packing of electrons in space, i.e. on ρ. Under normal circumstances in laboratory and in solar interior, electrons are very loosely spread over states with great majority of low states vacant. However, there are stars with structure very different from the Sun and other main sequence stars, in which because of extreme high density electrons are forced to occupy almost all states below E_F. Electrons in such a situation are said to be degenerate. Degeneracy affects distribution of momentum of electrons hence its contribution to pressure ought be accounted for.

In a co-moving system of coordinates centered at electrons most likely position $0 =< x >=< p_x >$ hence Δx and Δp_x are close to mean absolute values of distance between electrons d and their momentum p_x. For dense packing of electrons $\mu_e/\rho \sim d^3 \sim \Delta x \searrow$ their momenta must increase

$$p_x \sim \Delta p_x \sim h \left(\frac{\rho}{\mu_e}\right) \nearrow . \tag{5.14}$$

In calculation of pressure as flux of momentum one must account for different relation of momentum and velocity p_x and v_x for classical and relativistic motion of electrons:

$$P \sim \frac{\rho}{\mu_e} p_x v_x \sim \frac{\rho}{\mu_e} p_x \begin{cases} \dfrac{p_x}{m_e} & \text{classical} \\ c & \text{relativistic} \end{cases} \tag{5.15}$$

One has to be careful here to not confuse mass per electron μ_e (proton mass for H) and mass of electron itself m_e. Substitution of Eq. (5.14) yields

$$P \sim \begin{cases} \dfrac{h^2}{m_e}\left(\dfrac{\rho}{\mu}\right)^{5/3} & x \ll 1 \\ hc\left(\dfrac{\rho}{\mu}\right)^{4/3} & x \gg 1 \end{cases} \tag{5.16}$$

where $x \equiv p_x/m_e c \sim (h/m_e c)(\rho/\mu)^{1/3} \sim \lambda_C/d$ and $\lambda_C = h/m_e c$ is Compton wavelength of electron. Substitution of Eq. (5.14) yields critical density for classical and relativistic case $\rho_{rel} \sim \mu/\lambda^3 \sim \mu(m_e c/h)^3 \sim 10^6 g/cm^3$. The degeneracy pressure calculated in this way does not depend on temperature. It depends on density and corresponding momentum which by virtue of Pauli and Heisenberg principles may never be reduced below Fermi value.

Note that neutrons similarly to electrons have spin $1/2$ hence are fermions. They suffer from degeneracy in the similar way as electrons. The difference is that for neutrons μ_e and m_e both become μ_H-mass of a neutron. Another difference is role played by additional nuclear interactions, which render our formulae for neutrons rough approximation only.

Partial degeneracy

Fermi thermal energies of particles may be considered as obligatory and luxury forms of energy. Particle are free to exchange energy kT with their environment in such processes as cooling and heating. For sufficiently high densities this energy becomes small compared to the Fermi energy $kT < E_F = p_F^2/m_e$ and corresponding contribution of thermal motions to pressure becomes similarly small. The corresponding density of degeneracy may be computed from a condition

$$1 > \frac{kTm_e}{p_x^2} \sim \frac{kTm_e}{h^2}\left(\frac{\mu_e}{\rho}\right)^{2/3} \sim \frac{1}{500} \tag{5.17}$$

The corresponding density of degeneracy setting in is

$$\rho_{deg} \sim \left(\frac{kTm_e}{h^2}\right)^{3/2} \mu_e \sim 10^2 g/cm^3 \tag{5.18}$$

For thermal and Fermi energies comparable degeneracy becomes partial. It may be best understood by consideration of plots of occupation of states. At negligible temperature transition from full to zero occupation occurs at E_F sharp. For partial degeneracy this transition becomes blurred over an energy range of order kT.

5.1.5 Other high density effects

Cristallization

Electric charges of CNO ions in stellar interiors are large and tend to repel themselves vigorously. As soon as their electrostatic potential exceeds thermal energy,

$$\frac{Z^2e^2}{\left(\frac{\mu}{\rho}\right)^{1/3}} \sim 100kT \tag{5.19}$$

electrostatic forces affect ion distribution. In these circumstances ions regular arrangement of ions maximizes their distances minimizing energy. In effect ions form a cristall grid. By solving we obtain the corresponding critical density for this process

$$\rho \sim \frac{100kT}{Z^2e^2}\mu \geq 10^7 g/cm^3 \tag{5.20}$$

These are densities present nowhere in the main sequence stars. However they are reached in massive white dwarfs and neutron stars.

Neutronization

Simplest nuclear reactions converting barions are those involving neutrons and protons:

$$n \rightarrow p + e^- + \overline{\nu}_e \tag{5.21}$$

$$p + e^- \rightarrow n + \nu_e \tag{5.22}$$

$$\tag{5.23}$$

Because of conservation of charge and leptonic number these reactions involve electrons and neutrina. First reaction, a β^- decay of neutron, occurs spontaneously with release of energy and with half-time of decay 18 min. Conversion of proton into neutron in a process called electron capture consumes energy, hence it does not occur spontaneously under normal conditions. Particle physicists say that proton and neutron correspond to ground and excited states of a barion. This situation may be violated in conditions of extreme degeneracy forcing electrons to fill completely all states up to so high threshold Fermi energy E_F that $E_F + m_e c^2 > 1.3 MeV$, where $m_e c^2 = 0.5 MeV$ is electron's rest energy. Then combined energy of p and E_F of electron exceeds that of neutron, so that conversion of protons into neutrons occurs spontaneously. Free neutrons might attach to atomic nuclei, as they suffer from no electrostatic repulsion $(Z, A) + n \to (Z, A+1)$. Protons in atomic nuclei also capture electrons. In effect most matter is transformed into super massive nuclei with significant excess of neutrons, or neutronization sets in. The required critical Fermi energy of electrons exceeds the rest energy, hence electron motion is relativistic. Their corresponding momentum is $p_F \sim 2m_e c$. Heisenberg indeterminacy principle yields minimum volume occupied by an electron in the phase space: $(\mu/\rho)(2m_e c)^3 > h^3$. Solving we obtain the critical density for setting in of neutronization

$$\rho \sim 10^7 g/cm^3 \tag{5.24}$$

The corresponding separation of electrons is $d \sim (\mu/\rho)^{1/3} \sim \lambda_C/2$ where $\lambda_C = h/m_e c$ is Compton electron wavelength. At still higher densities, $\rho \sim 10^{11} g/cm^3$ excess of neutrons becomes so drastic that these nuclei decay into fluid of neutrons. Neutronization becomes complete by then.

5.2 Opacity

In stellar structure equations radiation transfer is treated in the diffusion approximation. In this approximation many interactions of matter with radiation are combined together into just one function of state, namely Rosseland mean opacity $\overline{\kappa}_R$ such that

$$\begin{aligned} \frac{1}{\overline{\kappa}_R}\frac{dB}{dT} &\equiv \int \frac{d\nu}{\kappa_\nu}\frac{dB_\nu}{dT} \quad \text{where} \\ \kappa_\nu &= \frac{1}{\rho}\sum_i \sigma_\nu^{(i)} n_i \end{aligned} \tag{5.25}$$

and summation extends over all atoms and ions present and σ_ν is atomic cross section for scattering and absorption. Planck distribution and its integral are denoted with B_ν and B, respectively. Contributing processes are roughly split into scattering and pure absorption. In scattering number of photons and their energies are preserved, only directions change. In pure absorption photons are destroyed only to be emitted much later and in a possibly unrelated form. In contrast to stellar atmospheres in stellar interiors contributions by scattering and pure absorption are simply added together. Pure absorption also called photoabsorption cause electrons which change their energy at the cost of absorbed photon. Electrons either are bound in atoms or roam freely. Depending on the type of initial and final states of electrons photoabsorption is classified as free-free, bound-free and bound-bound (f-f, b-f and b-b). The energy conservation laws corresponding to these 3 processes are the following

$$\text{f-f} \qquad h\nu = \frac{m_e v_2^2}{2} - \frac{m_e v_1^2}{2} \tag{5.26}$$

$$\text{b-f} \qquad h\nu = \chi_\infty + \frac{m_e v_2^2}{2} \tag{5.27}$$

$$\text{f-f} \qquad h\nu = \chi_2 - \chi_1 \tag{5.28}$$

$$\tag{5.29}$$

where $h\nu$ is photon energy, v_1 and v_2 are initial and final electron velocities, χ_∞, χ_1 and χ_2 are ionization energy and energies of electron initial and final bound states.

5.2.1 Scattering

Electromagnetic wave passing an electron acts on it with an oscillating electric force and thus induces its oscillations. Oscillating charge becomes source of a spherical wave. This is classical wave description of scattering of photons by free electrons. Since the original and scattered frequencies match, so do energies $h\nu$ of incoming and scattered photons. Remarkably enough for this problem classical and quantum analysis yield the same cross section for scattering (so called Thomson cross section). The corresponding opacity is

$$\kappa_{nu} = \frac{8\pi}{3}\frac{r_e^2}{\mu_e} \approx 0.2(1+X) \tag{5.30}$$

where r_e the classical electron radius satisfying $e^2/r_e \equiv m_e c^2$. Remarkably, the scattering cross section is independent of ν. This holds so long as scattering is non-relativistic, i.e. $h\nu \ll m_e c^2$, corresponding to $T < 10^8 K$.

5.2.2 Free-free absorption

Free-free absorption may be considered inellastic collision of photon and electron. Yet total momentum and energy must be preserved in the collision. No unique relation exists between total initial momentum and energy, as the result of vector addition of photon and electron momenta depends on their orientation. However, in the final electron energy and momentum is related in the unique way $E = p^2/2m_e$. This demonstrates that such a simple inellastic collision would violate either energy or momentum conservation laws. Free-free absorption may occur solely in presence of a charged third boy, e.g. an ion. Because of its large mass, an ion may absorb considerable amount of momentum without taking much energy. Hence its presence enables conservation of momentum and still may be neglected in energy considerations, so that the following simple energy balance (Eq. 5.30) holds. Given an electron-ion pair, cross section for absorption is $\sim Z^2\nu^{-3}$, a classical result in electrodynamics. The effective opacity is also proportional to the number of electron-ion pairs, i.e. to ρ^2 and to the average time of their brief encounters $\tau \sim 1/v \sim T^{-1/2}$, so that the opacity per unit volume, first derived by Kramers becomes

$$K_\nu = Z^2\rho^2 T^{-1/2}\nu^{-3} \tag{5.31}$$

To get Rosselland average opacity we ought replace ν with Planck distribution ν_{max} where $h\nu_{max} \sim kT$ according to Wien law, so that

$$\overline{\kappa}_{f-f} = 3.8 \cdot 10^{22}(1+X)(X+Y+B)\rho T^{-7/2}\nu^{-3} \tag{5.32}$$

where $B = \sum_{i>2} X_i Z_i^2/A_i$ and factors in brackets account for contribution of electrons and ions, respectively. The role of heavy ions of CNO groups becomes evident from the dependence of opacity on their charge Z^2.

5.2.3 Bound-free absorption

Bound -free absorption is selective in that absorbed photons must have at least sufficient energy to ionize an atom, i.e. $K_\nu = 0$ for $h\nu < \chi$ or $\nu < \nu_0 = \chi/h$. For the frequencies above ionization limit ν_0 the cross section is proportional to ν^{-3} in a way reminiscent the classical result for free electrons.

$$\sigma_\nu \sim \left\{ \begin{array}{cc} 0 & \nu < \nu_0 \\ \left(\frac{\nu}{\nu_0}\right)^{-3} & \nu > \nu_0 \end{array} \right. \tag{5.33}$$

Some quantum corrections, called Gaunt factors, apply but do not change much the classical result. Ionization from different ions and different energy levels produces different ionization limits, so that the combined opacity in the $\log \kappa_\nu - \log \nu$ plane has see-saw tooth profile, with vertical decline towards low frequencies and inclination -3 decline towards high frequencies.

For bound-free photoabsorption contribution of hydrogen dominates because of its large fractional content. At intermediate temperatures this is effected by photoionization of neutral H atoms. For temperatures as low as $T \sim 3000 - 5000K$ hydrogen in stars is forms negative ion H^- (a proton with 2 electrons). Its ionization energy is as low as $\sim 1eV$ hence at these conditions it dominates absorption of low energy thermal photons.

At still lower temperatures form simple chemical molecules e.g. oxides of most populous elements. In these conditions calculation of composition and opacity becomes extremely difficult and is known only very crudely.

5.2.4 Bound-bound absorption

Bound-bound transitions corresponding to jumps of electrons within atomic structure contribute to absorption only at discrete values of corresponding to difference of energy between corresponding levels. In these way spectral lines are formed. Spectral lines of H and He, by far the most populous elements in stars are few and loosely scattered in the spectra. Because of this reason they were neglected for many decades of years in calculations of stellar structure in favour of simple Kramers-like formulae for f-f and b-f absorption. Spectral lines were considered important either for low temperatures because of multiple molecular lines and for very broad lines produced at high densities by electrostatic perturbation of energy levels by nearby nuclei. Only as late as at 1990 efforts of two large groups of atomic physicists, OPAL at Lawrence Livermore Labs (US) and OP, an international collaboration headed by M. Seaton (Brittain) produced realistic tables of $b-b$ opacities. These were huge enterprizes involving calculations for many dosens of elements, hundreds of ions and milions of spectral lines. For many of these completely new atomic quantum models involving many wave functions for each energy level were calculated, as experimental data on most energy levels of ions are missing.

Results of both groups are consistent and rather surprising for astrophysicists. It turned out that contribution by milions of lines of metals particularly those of iron group (Fe, Ni, Co) dominate other sources of opacity in stars. These calculations revealed that previously opacity in stars was underestimated by a factor of more than 5. Although the new opacities did not invalidate old qualitative results of the stellar structure and evolution theory, consistency of quantitative results with observations markedly improved. In particular, dynamical estimates of stellar masses from binary stars and from their pulsation (dynamical) times are no longer in factor of 2 disagreement. In effect, Kramers opacity no longer is considered valid approximation for stars. In any case a formula of the type $\overline{\kappa} \sim \rho^{1/2}T^{-5/2}$ approximates true opacity better. In this way astronomers no longer are able to rely on simple opacity formulae. As calculation of opacity became highly complex and technical business beyond capabilities of individual researches, modern stellar structure calculations rely on interpolation of OPAL and/or OP tables. For typical chemical compositions these tables are available "off-shelf" via internet. For more exotic compositions authors of tables usually are willing to perform extra tabulation on request.

5.3 Energy production

5.3.1 Stellar sources of energy

Age of the Sun

Nuclear dating of radioactive isotopes fixes solar age $\tau_\odot$ at a value exceeding $4 \cdot 10^9 y$ and less than $5 \cdot 10^9 y$. The lower limit comes from content of lead in oldest uranium ores. It bases on plausible assumption that the Earth and its oldest rocks condensed no sooner than formation of the Sun. Uranium nuclei decay in a chain of reactions eventually producing lead, so older age means more uranium converted to lead. In a solid rock neither uranium nor lead atoms migrate, hence the ratio of their content yields age.

The upper limit comes from fractional content of ^{235}U and ^{238}U isotopes. Both isotopes are radioactive and decay. However, their half-times of decay are different. Hence the ratio of their content evolves with time. All nuclei heavier than Fe including uranium were produced in stellar cores from lighter nuclei by capture of free neutrons. In the process some of the neutrons are converted to protons to maintain equilibrium relative content of protons and neutrons. As result of stellar wind and supernova explosions these nuclei left stellar interiors and enriched interstellar matter. This must occur before condensation of a cloud of interstellar matter to form our Sun and its planets. In any case, experimentally measured cross sections for capture of neutrons enable prediction of the $^{235}U/^{238}U$ formation rate due to irradiation by free neutrons. Comparison of the original ratio with the presently observed isotope ratio yields age prior to formation of the Sun. During its life the Sun lost $E = \tau_\odot L_\odot$ of energy, or a fraction $E/M_\odot c^2 = \tau_\odot L_\odot / M_\odot c^2 \sim 0.0004$ of its rest mass energy.

Alternative sources of stellar energy

Discussion of several alternative sources of stellar energy reveals clearly difficulty in supplying enough energy for stars. Let us consider chemical reactions. Of these one of most efficient source of energy in synthesis of water by burning of hydrogen: $2H_2 + O_2 \to 2H_2O$. Synthesis of each particle of mass $18\mu_H$ yields $3eV$ of energy, i.e. 10^{-10} of the rest mass energy. This is far short to power the Sun during its life.

By virtue of the virial theorem, the total energy of a star is of order of its thermal energy and both are of order GM^2/R. The gravity energy of the Sun constitute only fraction of $GM^2/RMc^2 \sim GM_\odot/R_\odot c^2 \sim 10^{-6}$. Again this energy is not enough. In fact we already established that the thermal energy suffices to supply solar radiation only for the thermal time scale $\tau_T \sim 10^7 y$ (Eq. 2.24).

Although for ordinary stars the gravity energy is not efficient enough, it is extremely efficient source of energy for more compact bodies encountered in astrophysics, e.g. white dwarfs, neutron stars and black holes. For the latter the gravity energy at the horizon is Mc^2 exactly, hence it is 100 percent effective source of energy.

Nuclear binding energy

No alternative sources of energy considered yet proved efficient enough to power solar radiation. This forced astrophysicists to turn to nuclear sources of energy. Nuclear binding energy may be computed by calculation of the defect of mass of the nucleus compared to the summed mass of its component nucleons, i.e. protons and neutrons:

$$f \equiv \frac{E_B}{A} = \frac{(A-Z)\mu_n + Z\mu_p - M_{A,Z}}{A} \tag{5.34}$$

Defect of mass f (in energy units) as function of nuclear mass A is plotted in Fig. **??**. Its general features are well explained with the drop model of nucleus developed by the Kopenhagen school of physicists. The basic nuclear binding force acts on a short range, only between adjacent nucleons and is independent of of charge of nucleons. One may picture nuclear forces as a kind of glue binding nucleons. In light nuclei large fraction of nucleon surface is on the surface of a nucleus, i.e. it is not glued to anything. The resulting binding energy and mass defect are small. In massive nuclei most nucleons are surrounded from all sides by neighbors and their nuclear binding energy energy becomes saturated at a constant value. These heavy nuclei contain large number Z of protons. Their electrostatic repulsive force has long range so each proton repells all remaining ones. The total electrostatic energy is proportional to Z^2, corresponding to the decrease of the binding energy $\sim Z^2/A$. These two opposite effects produce maximum nuclear binding energy per barion in ^{56}Fe nucleus. Onto this general trend are superimposed substantial fluctuations in binding energy of consecutive nuclei. Their cause may not be explain within the drop model. They are thought to arise due to additional binding force arising due to pairing of like nucleons with opposite spins, separately for protons and neutrons and by filling complete nuclear shells. One may think of analogy with strong binding of complete atomic shells in noble gases.

In effect fusion of two light nuclei into a heavier one results in net growth of the binding energy per nucleon. Let us focus our attention on fusion of 4 protons into a helium nucleus He^4:

$$\frac{E_N}{Mc^2} \equiv \frac{4\mu_p - \mu_{^4He}}{\mu_{^4He}} \sim 0.007 \tag{5.35}$$

This efficiency of energy production is more than enough to power our Sun and most other stars. In fact by calculation of the thermal time scale we established that nuclear sources of energy in the Sun suffice for $10^{10}y$. Further chain of reactions converting ^4He into ^{56}Fe yields only 20% of E_N. Note that direct fusion of protons would produce nuclei with excess number of protons over neutrons compared to the stable nuclei. Hence fusion must be accompanied by reactions converting protons to neutrons, e.g. electron capture and/or β^+ emission of positons (positive charged electrons). These latter reactions affect charge of nucleus but not its number of nucleons.

Synthesis of heavy nuclei

For $A > 56$ fusion no longer yields energy and does not occur spontaneously. Heavy nuclei are produced by consecutive capture of a number free neutrons. Modest amounts of free neutrons are produced in ordinary stars as a byproduct of more exotic chains of reactions binding protons into helium. Lack of charge enables these neutrons to stick easily to heavy nuclei. For similar reasons as before, capture of neutrons must be followed by β^- decays to preserve stable ratio of protons to neutrons. Generally β^- decays occur in time short compared to the likely time of neutron capture. Hence this so called slow or s-process at each stage increases number of nucleons A by just one. After setting on of the equilibrium the same number of nuclei is produced at each stage per unit time, as excessive speed would result in dearth of original nuclei. Constant production rate means $\sigma_i n_i = const$, where σ_i and n_i is cross section for i-th reaction and density of its original nuclei. Reality of the s-process was confirmed by observation that for most nuclei cosmic abundance n_i is inversely proportional to the production cross section $n_i \sim 1/\sigma_i$.

There are however nuclei of mass A for which no stable nuclei with mass $A - 1$ exist. These nuclei may not be produced in the s-process. Their origin was understood only after it became clear that in outbursts of supernovae large fraction of stellar core is within seconds converted into neutrons. These neutrons have chance to stick to β unstable nuclei before they manage to decay. In this way capture of one or several neutrons leads to formation of a stable nucleus through one or more unstable nuclei. This process is called rapid or r-process of neutron capture.

5.3.2 Nuclear reaction rates

Tunelling

Although fusion of light atoms yields huge amount of energy, these reactions do not happen in ordinary circumstances. This is so for nuclei repel each other over long distance with large electrostatic force. For charges $+Z_1e$ and $+Z_2e$ and typical nuclear radius $r \sim 10^{-13}A^{1/3}cm$ the corresponding Coulomb potential barrier is $E_C \sim Z_1Z_2e^2/r \sim Z_1Z_2MeV$. This barrier must be compared with the typical thermal energy within stellar interiors $kT \sim 1keV$ for $T \sim 10^7K$. By virtue of Boltzmann distribution, the fraction of thermal atoms which may reach energies as large as E_C is $e^{-E_C/kT} \sim e^{-1000} \sim 10^{-430}$. Clearly, chances to have enough energy for a single atom in a galaxy are slim, not to mention just a star. Nuclear reactions would not occur in stars if not process of quantum tunelling of potential barriers.

In the world of particles energy is conserved only in sufficiently long time intervals. According to Heisenberg indeterminacy principle in very short time intervals Δt energy of a particle may not be constrained more than such ΔE that $\Delta E\Delta t \geq \hbar$. Thus for a particle it is quite probable to to acquire virtual energy in excess of the Coulomb barrier $E_C \sim e^2/r$ for a time interval not exceeding $\Delta t \sim \hbar/E_c \sim \hbar r/e^2$. However, even in such a short time energy may fluctuate both up and down, so probability $P_{\Delta t}$ of energy exceeding Coulomb barrier in interval Δt satisfies $P_{\Delta t} < 1/2 = e^{-0.69}$. In fact exact theory yields $P_{\Delta t} = e^{-2\pi}$.

Let typical width of a barrier is r or so. Typical flight time of a particle of velocity v across the barrier is $\Delta T \sim r/v$. This time is long compared to Δt. Hence successful penetration of Coulomb barrier demands fortuitous occurrence of $k \sim \Delta T/\Delta t \sim e^2/\hbar v$ consecutive fluctuations over the top of the barrier. The probability of such an event is

$$P_{\Delta T} \sim (P_{\Delta t})^k \sim e^{-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}} \tag{5.36}$$

where we accounted properly for nuclei charges Z_1 and Z_2. Finally, the cross section for the reaction is also proportional to the probability of reaction of nuclei after penetration of the barrier $S(E)/E$, where $E = mv^2/2$ and $m = m_1m_2/(m_1 + m_2)$ is so called reduced:

$$\begin{aligned} \sigma &= S(E)E^{-1}e^{-\frac{\overline{\eta}}{E^{1/2}}} \quad \text{where} \\ \overline{\eta} &= 2\pi\frac{m^{1/2}}{2}\frac{Z_1Z_2e^2}{\hbar} \end{aligned} \tag{5.37}$$

$E = mv^2/2$ and $m = m_1m_2/(m_1 + m_2)$ is so called reduced mass of particles. Here particular choice of the shape of function $S(E)$ is motivated by its smoothness. This smoothness facilitates interpolation and extrapolation of experimental cross-sections.

Reaction rates

The rate of reactions per unit time and volume between particles of type i and j and density n_i and n_j is proportional to the total number of pairs of particles and volume of a cylinder of cross section σ and length v swept by each particle while moving with velocity v.

$$r_i j = \frac{1}{1+\delta_i j} n_i n_j < \sigma v > \tag{5.38}$$

The factor $1/(1+\delta_{ij})$ assumes value 1 for different particles and $1/2$ for identical particles, accounting for double counting of them in pairs. The average swept volume is defined as

$$\begin{aligned} < \sigma v > &= \int \sigma(E) v f(E) dE \quad \text{where} \\ f(E)dE &= \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-\frac{E}{kT}} dE \end{aligned}$$

is Maxwell distribution of particle energies. Substitution of the cross-section and Maxwell distribution yields

$$< \sigma v >= \frac{1}{m^{1/2}(kT)^{3/2}} \int_0^\infty S(E) e^{-\frac{E}{kT} - \frac{\overline{\eta}}{E^{1/2}}} dE \tag{5.39}$$

Integration of $< \sigma v >$ involves a slowly varying function $S(E)$ and a rapidly varying exponential function with a sharp maximum at energy E_0. Such integrals are approximately evaluated by the steepest descent method. In this method the sharp maximum is approximated with a gaussian exponent centered at E_0 considered an approximation of Dirac δ function. Location of the maximum follows from differentiation of the exponent:

$$\begin{aligned} 0 &= \frac{d}{dE}\left(-\frac{E}{kT} - \frac{\overline{\eta}}{E^{1/2}}\right) \quad \text{hence} \\ E_0 &= \left(\frac{1}{2}\overline{\eta} kT\right)^{2/3} \end{aligned}$$

Second derivative of the exponent is required to compute width of the gaussian function. Evaluation of the corresponding gaussian integral yields

$$\begin{aligned} < \sigma v > &= \frac{S(E_0)}{m^{1/2}(kT)^{1/2}} \tau^{1/2} e^{-\tau} \sim \tau^2 e^{-\tau} \quad \text{where} \\ \tau &= 3\frac{E_0}{kT} \end{aligned} \tag{5.40}$$

Defining $W \equiv Z_1^2 Z_2^2 A \equiv Z_1^2 Z_2^2 A_1 A_2/(A_1 + A_2)$ one obtains $\tau \approx 20 W^{1/3} T_7^{-1/3}$.

Our derivation demonstrates that the reaction rates are exponential functions of temperature. However in some applications these rates are approximated by power functions of temperature. The exponent of approximating power function may be calculated as

$$\begin{aligned} n &= \frac{dln < \sigma v >}{dlnT} \approx -\frac{2}{3} - \frac{d\tau}{dlnT} \approx -\frac{2}{3} - \tau\frac{dln\tau}{dlnT} \quad \text{where} \\ \epsilon &\sim \epsilon_0 \rho T^n \end{aligned} \tag{5.41}$$

Burning of hydrogen to helium occurs in two alternative reaction chains, so called $p - p$ cycle and CNO cycle. Accordingly two approximate expressions arise for the reaction rate exponents:

$$n \approx -\frac{2}{3} - \frac{\tau}{3} \approx \begin{cases} 6 & \text{for} \quad p - p \\ 20 & \quad\quad\;\; CNO \end{cases} \tag{5.42}$$

for $T = 10^7 K$.

For large densities the reaction rates are faster due to effective lowering of Coulomb barrier by screening of nuclear charges by electrons forced in their vicinity.

5.3.3 Reaction cycles

Nuclear reactions were considered source of stellar energy right from their discovery. By far most important nuclear reactions in stars are these which convert most abundant fuel, hydrogen, into helium. Details these reactions were revealed by Bethe and von Weizsäcker in late 1930's. Namely, hydrogen is burned into helium in two separate cycles of reactions called p-p and CNO cycles.

P-p cycle

The p-p cycle starts from a proton p. The reactions may be described as fusion with another proton immediately followed by a β^+ decay and another fusion with a proton. In this way are produced ^3He nuclei. Fusion of 2 ^3He nuclei yields the final product ^4He. They could be remembered with the aid of the following mnemonic notation:

$$p \quad p\beta p \quad 2\,^3He \tag{5.43}$$

Production in these reactions of other particles follows from requirements of conservation of charge e^+, of number of leptons e^+ and ν, and of number of nucleons $2p$. In particular, leptons are light particles, such as electrons and neutrinos. They interact with nucleons only by means of so called weak nuclear force. The nature of the weak force is as different from ordinary (strong) nuclear force as electromagnetic force is different from the gravity one. All four forces constitute the complete set of basic types of interactions in physics. Each lepton in the total lepton number counts as $+1$. Their antiparticles, positive electron (positon) and antineutrino, e^+ and $\overline{\nu}$ count as -1. Nucleons p and n count as 0 leptons. To conserve leptons their total number on the left and right hands of reactions must agree. All involved reactions are written in full in Fig. **??**. Note in particular, that apart from the described chain of reactions identified as $pp1$, there are two other chains called $pp2$ and $pp3$. These additional chains play lesser role in the energy production as they account for no more than 31 and 0.1 % of reactions, respectively. Importance of the $pp3$ cycle lays in production of neutrinos which are easily detectable due to their peculiar energy and thus may be used for diagnostic of the solar thermonuclear reactor.

Note that no stable ^2He nucleus exists. Hence the p-p fusion and β^+ decay must occur simultaneously in order to produce a stable product - the deuterium ^2H. This means that a weak interaction must occur during brief collision of two protons, lest they separate again. True to its name the cross section for weak interaction in first reaction of the p-p cycle is extremely small. This reactions may occur only because of unusual density of hydrogen in stars.

CNO cycle

In the CNO cycle the ^{12}C nucleus acts as a work bench on which a ^4He nucleus is assembled in several operations. these operations may be noted mnemonically as

$$^{12}C \quad p\beta pp\beta p \quad 2\,^3He \tag{5.44}$$

Note that in both p-p and CNO cycles intermediate nuclei suffer from excess of protons, hence all β decays are the β^+ decays emitting positons. In the final reaction ^4He nucleus is separated from ^{12}C. Again the involved reactions are written in full in Fig. **??**. In particular note extension of the CNO cycle up to ^{17}O. In contradiction to the p-p cycle all fusion reactions of the CNO cycle involve strong nuclear interactions only.

Relative role of the p-p and CNO cycles

At high temperatures the CNO cycle dominates. Then energies of the colliding particles are large and Coulomb barrier poses no important obstacle. Cross sections for all fusion reactions in the cycle are large, for all involve strong nuclear interactions between nucleons. Leptons in the CNO cycle are produced only via β decays. The p-p ccycle at this temperatures is still active, but due to low probability of the first weak reaction p-p it poses no competition for the other cycle. In low temperatures Coulomb barrier becomes serious obstacle in tunelling, as cross section falls exponentially. In this case large charge of nuclei reacting in the CNO cycle becomes serious obstacle for tunelling and its efficiency falls dramatically. So at low temperature remaining reactions are those of the p-p cycle. Though its first weak reaction is slow, Coulomb barier for light nuclei poses less of a problem.

Advanced thermonuclear burning

Hydrogen burning constitutes basic source of energy for all homogeneous stars, i.e. stars on the main sequence. Other reactions start playing any role only after exhaustion of hydrogen in the stellar core. Helium burns at higher temperature than hydrogen in a so called 3α reaction $3\ {}^4He \rightarrow^{12} C$. Direct collision of 3α particles and their reaction is extremely unlikely even in dense cores of stars. Hence the 3α reaction runs in two stages. In first stage 2 α particles yield an unstable ${}^8Be$ nucleus. Normally after a while ${}^8Be$ splits again into 2 α particles. The 3α reaction succeeds only in these cases when a 3-rd α particle manages to attach to 8^Be during its brief existence:

$$\begin{array}{lcll} {}^4He+{}^4He & \leftrightarrow & {}^8Be & -0.095MeV \\ {}^8Be & \rightarrow & {}^{12}C+2\gamma & +7.6MeV \end{array} \quad 3\alpha \tag{5.45}$$

Further reactions involving fusion of α particles convert

$${}^{12}C \rightarrow {}^{16}O \rightarrow {}^{20}Ne \cdots . \tag{5.46}$$

At still higher temperatures burns carbon

$$\begin{array}{lcl} {}^{12}C+{}^{12}C & \rightarrow & {}^{23}Na+p \\ & \rightarrow & {}^{20}Ne+\alpha \end{array} \quad 3\alpha \tag{5.47}$$

and follow up reactions.

Reactions producing free neutrons are of considerable importance for s-process synthesis of heavy nuclei. Examples of such reactions are

$$\begin{array}{lcl} {}^{13}C+{}^4He & \rightarrow & {}^{16}O+n \\ {}^{22}Ne+{}^4He & \rightarrow & {}^{25}Mg+n \end{array} \quad 3\alpha \tag{5.48}$$

The target nuclei are produced in the CNO cycle either directly, ${}^{13}C$ or by sequence of 2 fusion of ${}^{14}N$ with α particles.

5.3.4 Neutrino losses

Losses in ordinary reactions

Discussion of hydrogen burning cycles revealed that weak interactions lead to production in nuclear reactions of neutrinos and/or antineutrinos. Neutrinos interaction with matter is so extremely weak, that great majority of them leaves star with no further interaction with its matter. With them they take a fraction of energy released in thermonuclear fusion. This loss of energy is accounted for by lowering yield of nuclear reactions E_{ij} (Eq. 4.48).

Urca processes

There are pairs of nuclear reactions which produce neutrinos without affecting chemical composition.

$$\begin{array}{lcl} (Z,A)+e^- & \rightarrow & (Z-1,A)+\nu \\ (Z-1,A) & \rightarrow & (z,A)+e^-+\overline{\nu} \end{array} \quad 3\alpha \tag{5.49}$$

The only effect of these reactions is loss of energy carried away by the neutrinos. Such reactions require high densities and at least partial degeneracy for efficiency of the initial electron capture. They are called Urca processes after a casino in in Rio de Janeiro where according to observation of nuclear physicists money were lost as quickly as neutrino energy from stars. Example are reactions involving ^{35}Cl and ^{35}S. Urca processes are responsible for cooling of cores of evolved stars. Their analogue involving neutrons and protons is responsible for cooling of neutron stars.

Chapter 6

Structure of homogenous stars

Stars are made of clouds of interstellar matter. The interstellar matter is rather well mixed, hence initial chemical composition of stars is homogeneous. Homogeneous stars made of perfect gas form the main sequence (MS) in the color-magnitude diagram (H-R diagram). The ideally homogeneous stars are called zero-age main sequence (ZAMS). Because of their age, our Sun and most other ordinary stars on the main sequence have burned already several percent of hydrogen in their center. So long as hydrogen burns vigorously in their center the MS stars do not differ much from the ZAMS ones.

6.1 Estimated parameters

Let us start from simple estimates. For this purpose we assume constant opacity, use virial theorem and replace derivatives in the continuity and radiative transport with ratio of variables. In this way we obtain the following set of algebraic equations:

$$\begin{aligned} \kappa &= const \\ kT &\sim \frac{GM\mu}{R} \\ \rho &\sim \frac{M}{R^3} \\ L &\sim R^2T^3\frac{T}{R}\frac{1}{\rho\kappa} \end{aligned}$$

Substitution into the last equation yields

$$\boxed{L \sim} R^2\frac{M^3}{R^3}\frac{M}{R^2}\frac{R^3}{M} \sim \boxed{M^3} \tag{6.1}$$

By fortunate coincidence exponents of R cancelled so that a solution was obtained without consideration of energy production. The present solution is not particularly realistic in assuming constant κ and radiative transport in the whole star. Nevertheless steep growth of luminosity with mass is roughly consistent with observations. The true exponent is closer to 4.

6.2 Homologous main sequence

6.2.1 Homlogous solutions

So far we obtained rather crude analytical solutions of stellar structure equations by replacing derivatives with finite differences. Similar solutions may be obtained in a more formal way as homologous solutions. For the main sequence models we assume homogeneous composition and perfect gas equation of state. It is convenient here to use dimension-less variables

$$\begin{aligned} x &= \frac{r}{R} \\ q &= \frac{m}{M} \\ f &= \frac{l}{L} \\ t &= \frac{T}{T_0} \\ p &= \frac{P}{P_0} \end{aligned}$$

where $T_0 = GM\mu/kR$ and $P_0 = GM^2/4\pi R^4$. Substituting the new variables into equations of stellar structure and using perfect gas equation of state, Kramers-like opacity law $\kappa = \kappa_0 \rho^\alpha T^{-\beta}$ and energy production $\epsilon = \epsilon_0 \rho T^\nu$ one obtains

$$\frac{dq}{dx} = \frac{p}{t} x^2 \tag{6.2}$$

$$\frac{dp}{dx} = -\frac{p}{t}\frac{q}{x^2} \tag{6.3}$$

$$\frac{dt}{dx} = -C \frac{f}{x^2} \frac{p^{\alpha+1}}{t^{\alpha+\beta+4}} \quad \text{for radiative or} \tag{6.4}$$

$$p = Et^{5/2} \quad \text{convective transport} \tag{6.5}$$

$$\frac{dt}{dx} = Dp^2 t^{\nu-2} x^2 \tag{6.6}$$

where

$$C = C_0 \frac{\kappa_0}{\mu^{\beta+4}} \frac{LR^{\beta-3\alpha}}{M^{\beta+3-\alpha}} \tag{6.7}$$

$$
\begin{aligned}
D &= D_0 \epsilon_0 \mu^\nu \frac{M^{\nu+2}}{LR^{\nu+3}} && (6.8)\\
E &= 4\pi K G^{3/2} \left(\frac{\mu}{k}\right)^{5/2} M^{1/2} R^{3/2} && (6.9)\\
KT^{5/2} &= P && (6.10)\\
KT_e^{5/2} &= P_e = \frac{GM}{R^2 \kappa} && (6.11)
\end{aligned}
$$

The last equation was obtained from the surface boundary condition $1 = H\rho_e\kappa$.

Let for a particular choice of C and D or E and D for radiative and convective models, respectively, exists a solution of the corresponding subsystem of four among five equations Eqs. (6.2) – (6.6). Such a solution may be obtained e.g. by numerical integration. Than this solution describes a whole family of stellar models with parameters M, R, L scaled in such a way that C, D or E, D remain constant. Such a family of models is called homologous models. Shapes of solutions for homologous models are the same except for different units.

Substantial information on properties of the homologous models may be obtained without solution of Eqs. (6.2) – (6.6). Elimination of L from Eqs. (6.16) and (6.18) yields

$$
const \equiv CD \sim R^{\beta-3\alpha-\nu-3} M^{\nu+2-\beta+\alpha-3} \qquad (6.12)
$$

For Kramers opacity $\alpha = 1$ and $\beta = 3.5$. However, modern opacity calculations demonstrate that Kramers formula is rather poor approximation and more appropriate one is $\alpha = 1/2$ and $\beta = 2.5$. We stick to the latter values.

6.2.2 CNO main sequence

For CNO nuclear burning $\nu \approx 16$. Then Eq. (6.12) yields

$$
\begin{aligned}
R &\sim M^{-\frac{\nu+2-\beta+\alpha-3}{\beta-3\alpha-\nu-3}} \sim M^{0.7} && (6.13)\\
T_0 &\sim \frac{M}{R} \sim M^{0.3} && (6.14)\\
& && (6.15)
\end{aligned}
$$

Thus for the main sequence stars the central temperature increases with mass. We already indicated that CNO cycle burns at high temperatures. Hence we conclude that CNO burns in the massive MS stars. For the upper main sequence (Sp: G0 – O5) observations of binary stars yield $R \sim M^{0.8}$ in rough agreement with our value. Solution of Eq. (6.16) for L yields

$$
L \sim R^{3\alpha-\beta} M^{\beta+3-\alpha} \sim M^{4.3} \qquad (6.16)
$$

$$
\begin{aligned}
T_e^4 &\sim \frac{L}{R^2} \sim M^{2.8} \qquad (6.17) \\
T_e &\sim M^{0.7} \qquad (6.18) \\
L &\sim T_e^{6.0} \qquad (6.19)
\end{aligned}
$$

The corresponding observed relations for the upper main sequence are $L \sim M^{3.6}$ and $L \sim T_e^{7.0}$ (Allen, 1973).

6.2.3 P-p main sequence

Similar considerations for the p-p cycle $(\nu \approx 4)$ yield

6.3 Results of MS calculations

Chapter 7

Stellar evolution

7.1 Red giants

7.2 Dynamics of stellar clusters

7.2.1 Dynamic equilibrium of clusters

Clusters of stars provide an important test of the stellar structure and evolution theory. Let us consider a cluster of N stars of mass M each of total radius R. Similarly to atoms in stars, average stellar velocity V in cluster in balance obeys virial theorem

$$\frac{G(NM)^2}{R} \sim NMV^2 \tag{7.1}$$

$$V \sim \sqrt{\frac{GNM}{R}} \tag{7.2}$$

Analogue of a dynamic time scale of a cluster is its crossing time

$$\tau_{cross} \sim \frac{R}{V} \sim \sqrt{\frac{R^3}{GNM}} \tag{7.3}$$

7.2.2 Relaxation time

Stars may change energy in result of interaction with another star. The average time of this happening is called the relaxation time τ_{rel}. After this time most of stars in the cluster interacted and established an distribution of velocities close to the equilibrium distribution, i.e. Maxwell distribution. Relative to the general distribution of stars, stars in clusters are densly packed, but still distances between them are huge compared to their size. Hence direct collisions of stars in clusters are extremely rare. However, even when they pass each other at a fairly large distance, say b, their mutual gravitational pull might be strong enough to change their direction of motion by a large angle if the gravity. For a star with kinetic energy MV^2 the cross section for interaction b^2 is derived from

$$\frac{GM^2}{b} \sim MV^2 \quad \text{hence} \tag{7.4}$$

$$b^2 \sim \frac{G^2M^2}{V^4} \tag{7.5}$$

We neglect factors of order of one following from Keplerian bending of trajectories and use straight lines in estimation of the minimum distance b. Similarly, we substitute the initial velocity V for the actual stellar velocity. For likely occurance of a mutual interaction, a star must cover such a distance Λ, called mean-free-path distance, over which its crosssection swept the volume Λb^2 containing one other star, on average. Assuming near constant density of stars over cluster volume R^3,

$$\frac{\Lambda b^2}{1} \sim \frac{R^3}{N} \tag{7.6}$$

$$\Lambda \sim \frac{R^3}{b^2N} \sim \frac{R^3V^4}{G^2M^2N} \tag{7.7}$$

Time required to cover distance Λ is the relaxation time:

$$\tau_{rel} \equiv \frac{\Lambda}{V} \sim \frac{RN}{V} \sim N\tau_{cross} \tag{7.8}$$

Note that in the calculation of the net gravity force acting on a star in Eq. (7.4) we accounted only for action of a star in the distance in the range, say $b < r < 2b$. However, for $l > b$, there are l^2 more stars in the distance range $l < r < 2l$ which gravity force is factor $1/l^2$ less, hence their combined pull is comparable to those in the distance b. In general $b < l < R$. Hence to account for distant interaction, the gravity force should be multiplied by $\log_2(R/b)$. As we demonstrated, this term is proportional to $\ln N$. With the appropriete correction, the exact expression for the relaxation time becomes

$$\tau_{rel} \sim 0.1\frac{N}{\ln N}\tau_{cross} \tag{7.9}$$

7.2.3 Evaporation of clusters

The dynamical arguments for young age of open clusters rely on consideration of the rate at which stars escape from clusters. Stars escape from clusters, or evaporate, when as result of interaction with other stars they acquire velocities in excess of the escape velocity V_{esc}. In the point-mass potential, the escape velocity or second cosmic velocity is by factor $\eta = \sqrt{2}$ larger than the typical orbital velocity. In clusters this factor is close to $\eta = 2$. For a maxwellian distribution of velocities in a cluster, a fraction $\gamma(\eta) \sim 0.01$ stars acquires velocities sufficient for escape. These stars escape within the relaxation time. Thus typical time for evaporation of the whole cluster is

$$\tau_{evp} \sim \frac{\tau_{rel}}{\gamma} \sim 100\tau_{rel} \tag{7.10}$$

7.2.4 Open and globular clusters

There are two kinds of clusters, namely globular clusters and open clusters. The globular clusters have near perfect spherical symmetry, contain of order of $N \sim 10^5$ stars packed densly, with a marked core. Their typical sizes and crossing times are of order $R \sim 10^{19.5}$ and $\tau_{cross} \sim 10^5$, respectively. Their highly regular shape would arise quite naturally for age older than the relaxation time $\tau_{rel} \sim 10^8$ yr. Their actual age is belived to be close to the age of universe of 10^{10} yr.

Open clusters contain of order of $N \sim 10^{2.5}$ stars forming a loose swarm, often of irregular shape. Their typical sizes and crossing times are of order $R \sim 10^{19}$ cm, and $\tau_{cross} \sim 10^6$ yr, respectively. Irregular shape might indicate age less than the relaxation time $\tau_{rel} \sim 10^6$ yr. They would completly evaporate if older than 10^8 yr. Hence, open clusters must be several orders younger than the universe.

Further indication of the old age of the globular clusters and young age of the open ones come from consideration of their distribution in the Galaxy, chemical composition and presence of interstellar matter and newly formed stars. We do not rely on them here to avoid circular arguments.

7.3 Evolution of star clusters

Evolutionary status of stars in clusters is best visualized on the color - magnitude diagrams. In honour of inventors, Hertzsprung and Russell, they are called the H-R diagrams. Both color and magnitude of stars are easily observable. They are simply related to important theoretical parameters, namely to the surface temperature and luminosity, easing comparision of observations with theory. The H-R diagrams for open and globular clusters look quite differently. The diagonal clustering of stars in the lower part of the H-R diagram corresponds to the main sequence. For open clusters the main sequence extends up to fairly luminous stars, called blue giants. Remaining few stars up and right of the main sequence are red giants. The main sequence of globular clusters ends much sooner with a sharp bend towards giant branch. The bend occures for relatively faint red stars.

These facts are best understood by consideration of nuclear time scale of stars on the main sequence. Let us adopt an approximate mass-luminosity relation for the main sequence: $L \sim M^{3.5}$. Since $\tau_n \sim M/L \sim M^{-2.5}$, massive star, i.e. those at the top of the main sequence (corresponding to large L) have shortest nuclear time scales. While the cluster ages, these stars first exchaust their fuel in the core and move on the H-R diagrm towards giants. Next slightly less massive stars leave the main sequence and so on. Hence the nuclear time scale of the star remaining on the very top of the main sequence just before its bend corresponds to the age of the whole cluster. In this way we see that the theory of stellar evolution predicts open clusters beeing young and globular cluster beeing old. The latter cluster lived long time till their faint and low mass stars were able to evolve away from the main sequence. This argument constitutes major test of the evolution theory.

Chapter 8

Terminal stages of evolution

8.1 Classification of terminal stages

Terminal stages of stellar evolution, in order of their increasing mass are listed in Table 8.1. In stars of insufficient mass central temperature never becomes high enough to ignite thermonuclear reactions. Thus they finish initial contraction as cool brown dwarfs. Since their thermal time scale is longer then age of the universe, their thermal cooling is negligible and their physical state does not change any more. Other stars remain on the main sequence for time of order nuclear time scale, then exhaust their hydrogen fuel in the core and become red giants. At this stage they often loose mass hence their final mass need not be the same as the initial mass. For this reason we list both. Many stars loose mass through intense stellar wind or ejection of envelope. Wind is revealed by characteristic distortion of stellar absorption lines. Ejected envelope forms a planetary nebula, which might me observed before its final dispersion. Cores of giants of sufficiently low mass are stable. After ejection of their hydrogen rich envelope and some cooling bare core forms hudrogen or carbon/oxygen white dwarf, the latter for slightly higher mass. Above $1.44 M_\odot$ white dwarfs are unable to support own weight. Following breaking-in or collapse of stellar core under its own gravity releases copious emounts of gravity and thermonuclear energy. Collapse increases both temperature and density hence thermonuclear reactions accelerate furiously. This energy ejects large part of the initial star and is observed as spectacular supernova outburst. If remnant has sufficiently low mass it is able to support itself. As nuclear reactions convert most of its content into neutrons it becomes neutron star. More massive remnants are unable to resist gravity and collapse further till become a black hole. At this point outside world may sense none of its properties except for mass and angular momentum effects influencing its gravity field. Note that existence of maximum mass for a neutron star is well established by theory,

Table 8.1: Terminal stages of stellar evolution

Name	$M_{init}/M_\odot$	$M_{fin}/M_\odot$
brown dwarf	< 0.08	< 0.08
white dwarf	< 10	< 1.4
neutron star	$< 30 :$	$< 4 :$
black hole	$> 30 :$	> 4

but calculation of its exact value is marred by dependence on admixture of other nuclear particles. Related uncertainity in masses is indicated by the colon.

8.2 Degenerate stars

8.2.1 Degenerate star in hydrostatic equilibrium

Depending on density pressure of degenerate electrons is expressed by two asymptotic formulae valid for non-relativistic and relativistic velocities:

$$P_d \sim \begin{cases} \frac{h^2}{m}\left(\frac{\rho}{\mu_e}\right)^{5/3} & \rho \ll \\ hc\left(\frac{\rho}{\mu_e}\right)^{4/3} & \rho \gg \end{cases} \left(\frac{mc}{h}\right)^3 \mu_e \tag{8.1}$$

where m, μ_e denote respectively electron mass and average mass per electron. Note that the inequality conditions on density ρ correspond to the follofing conditions on the electronic momentum: $p \ll\gg mc$.

The hydrostatic pressure is

$$P_h \sim \frac{GM^2}{R^4} \sim GM^{2/3}\rho^{4/3} \tag{8.2}$$

where we substituted $\rho \sim M/R^3$. Balancing two pressures we obtain two mass radius relations for non-relativistic and relativistic degeneracy:

$$\boxed{M} \sim \frac{h^6}{m^3\mu_e^5 G^3}\boxed{R^{-3}} \quad M \ll M_{Ch} \tag{8.3}$$

$$\boxed{M} \sim \frac{\left(\frac{ch}{G}\right)^{3/2}}{\mu_e^2} \equiv \boxed{M_{Ch}} \sim 1.44 M_\odot \tag{8.4}$$

The former equation yields opposite exponent of R to that for a constant density star. The latter equation for all radii yields the same mass, meaning that the configuation is in neutral equilibrium. This critical mass is called Chandrasekhar mass and is denoted with M_{Ch}. For $M > M_{Ch}$ P_h and P_d scale in the same with

Table 8.2: Properties of white dwarfs

Name	$M/M_\odot$	$R/R_\odot$	$c\frac{\Delta\lambda}{\lambda}$	$\frac{GM}{Rc}$	$M^{1/3}R$
Sirius B	1.05	.0074	90	90	.0075
40 Eri B	0.48	.0124	24	25	.0097
Stein 2051	0.50	.011		29	.0087
	0.7:			40:	.0097

density and always $P_h > P_d$ so that no degenerate star may exist in equilibrium. The value of M_{Ch} was calculated for pure helium white dwarf $(X = 0)$ using mass per electron $\mu_e \sim 1/(X + 0.5(Y + Z)) \sim 1/2$.

8.2.2 White dwarfs

Discovery

Sirius A is the brightest star on the sky. Its companion, Sirius B was discovered from perturbations to the streight line proper motion of Sirius A. These perturbations were consistent with Sirius B following keplerian orbit around common mass center with an unseen companion. By end of XIX century improved optics revealed image of Sirius B with luminosity by factor 10^5 less than Sirius A. Spectra revealed temperature (and thus surface brightness) of Sirius B higher than that of Sirius A. Thus for so low luminosity $L \sim 4\pi R^2 \sigma T^4$, radius Sirius B had to be at least factor $\sqrt{(10^5)} = 10^2.5$ less than that of Sirius A. This corresponded to volume less by a factor of 10^7. As motion of stars around mass centre indicated comparable masses, density of Sirius B had to exceed $10^6 g/cm$, indicating a star build of matter of yet unknown properties. This kind of stars were called white dwarfs. This puzzle of structure of white dwarfs was solved by Chandrasekhar who postulted a star supported by pressure of degenerate electrons.

Observed properties

Numerous white dwarfs are detected in imaging and proper motion survays as low luminosity blue spectrum stars. However reliable information may be obtained only for members of binary systems, e.g. such as listed in Table 8.2. In the table we list white dwarf mass derived from motion around mass center of the binary and radius derived from luminosity and spectral type or (σT^4). In the last column we list their combination $M^{1/3}R$. For non-relativistic degeneracy values in this column should be constant. Deviation for Sirius B may be explained by its high mass and resulting partially relativistic degeneracy.

Gravity redshift

Because of small size, gravity at surface of white dwarf is very strong. This enabled observation in white dwarf spectra of a relativistic effect called gravity reddening. Photon energy $E = h\nu$ according to general relativity corresponds to its dynamic mass of $m = E/c^2 = h\nu/c^2$. Note that photon rest mass is 0, so that photon of null energy has also null mass i.e. prectically does not exist. A photon with dynamic mass moving up in the gravity field looses energy as any other particle. Difference of initial and final energies depends on the gravity potential well:

$$\begin{aligned} h\nu_0 &- h\nu_\infty = \frac{GM}{R}m = \frac{GM}{R}\frac{h\nu}{c^2} \quad \text{hence} \\ c\frac{\Delta\nu}{\nu} &= \frac{GM}{Rc} \end{aligned} \tag{8.5}$$

Now, $c\Delta\lambda/\lambda$ is observable and may R if M is known from the binary motion or from $M - R$ relation. In this way gravity reddening may be exploited in verification of white dwarfs $M - R$ relation by observations. For this purpose we list in Table 8.2 both observed $c\Delta\lambda/\lambda$ and corresponding GM/Rc calculated from previous columns. Note good aggreement of observation with theory.

8.2.3 Neutron stars

By replacing electron mass with the neutron mass $m_n = \mu_n$ the same derivation holds approximately for neutron stars. In this way we obtain

$$\boxed{M} \sim \frac{h^6}{m_n^8 G^3}\boxed{R^{-3}} \quad M \ll M_{OW} \tag{8.6}$$

$$\boxed{M} \sim \frac{\left(\frac{ch}{G}\right)^{3/2}}{m_n^2} \equiv \boxed{M_{OW}} \sim 4 : M_\odot \tag{8.7}$$

$$\boxed{R_n} \sim \frac{m_e}{m_n} R_{WD} \sim 10km \tag{8.8}$$

In case of neutron stars present approximation becomes crude as it neglects effects of general relativity and role of other particles in the star. Oppenheimer and Wolkoff first considered these effects, hence the critical mass is somtimes designated as M_{OW}.

8.2.4 Pulsars

Because of very small surface, optical thermal radiation of neutron stars is very weak, even at very high temperatures. So discovery of neutron stars had to await observers at other bands, namely radio and X-ray. By the start of 1970's in some X-ray and radio sources, baptised pulsars were observed regular pulsations with sub-second periods. These pulsations turned out very stable, almost as stable as atomic clocks. For these reason they were connected with stellar rotation. However, one of the fasted pulsars discovered early, in the Crab nebula had period as short as $P \sim 0.03s$. In order to avoid disruption by the centrifugal force, its size had to be below the limit of

$$R < (P^2 GM)^{1/3} \sim 500km \tag{8.9}$$

No known star had as small radius as that. Hence observers turned towards neutron stars, as possible explanation.

8.3 Black holes

Chapter 9

Stellar pulsation

9.1 Eigenmodes

9.2 Pulsating variable stars

9.3 Excitation of pulsations

9.4 Pulsation mechanism

Chapter 10

Proto-stars

10.1 Stellar birth

10.2 Contraction of clouds

10.3 Evolution of proto-stars

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